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THE UNIVERSITY OF ALBERTA

ON THE ASYMPTOTIC NORMALITY OF A LINEAR COMBINATION OF
INDUCED ORDER STATISTICS WITH APPLICATION TO A CONDITIONAL
QUANTILE ESTIMATOR

by



SURYA PRAKASH UPADRASTA

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH
IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF STATISTICS AND APPLIED PROBABILITY

EDMONTON, ALBERTA

FALL, 1986

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled ON THE ASYMPTOTIC NORMALITY OF LINEAR COMBINATION OF INDUCED ORDER STATISTICS WITH APPLICATION TO A CONDITIONAL QUANTILE ESTIMATOR Submitted by SURYA PRAKASH UPADRASTA in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Mathematical Statistics.

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Dedicated to
my parents Mr. and Mrs. R.S. UPADRASTA
and
my ~~uncle~~ Mr. S.N. GODAVARTI

ABSTRACT

Let (X_i, Y_i) , $i=1, \dots, n$ be independent and identically distributed random variables with a bivariate continuous distribution function $H(x, y)$, $-\infty < x, y < \infty$. Let $X_{(1:n)} \leq X_{(2:n)} \leq \dots \leq X_{(n:n)}$ be the order statistics of X - observations and let $Y_{[1:n]}, Y_{[2:n]}, \dots, Y_{[n:n]}$ denote, respectively, the Y -observations associated with the above ordered $X_{(i:n)}$'s. The $Y_{[i:n]}$'s have been referred to in the literature as "induced" or "concomitants of" order statistics. In this thesis we are concerned with the asymptotic normality of linear combinations of induced order statistics with double weights, namely, weights which depend both on the ranks of X 's as well as the ranks of Y 's. This work extends the results of Stigler (1974) and Yang (1981). The method employed is the "Projection method" introduced by Hajek (1968).

We consider statistics of the form

$$S_n = n^{-1} \sum_{i=1}^n J\left(\frac{i}{n}, \frac{R_{in}}{n}\right) Y_{[i:n]}$$

where $J(u, v)$, $0 \leq u, v \leq 1$, is a suitable bivariate score function satisfying certain regularity conditions (see Theorem 3.1.1). Our results specialize to those of Stigler (1974) if we set $J\left(\frac{i}{n}, \frac{R_{in}}{n}\right) = J\left(\frac{R_{in}}{n}\right)$ and to those of Yang (1981) if $J\left(\frac{i}{n}, \frac{R_{in}}{n}\right) = J\left(\frac{i}{n}\right)$. In Part I of this thesis, it is proved that, under the regularity conditions referred to above,

$$P\left(\frac{S_n - ES_n}{\sigma(S_n)} \leq t\right) \rightarrow \Phi(t), \quad -\infty < t < \infty \text{ as } n \rightarrow \infty, \text{ where } \Phi(t) \text{ is the}$$

standard normal distribution function. The approach developed in Part I is then used in Part II to establish the asymptotic normality of a conditional quantile (Kernel) estimator proposed by Mehra (1986).

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CHAPTER I

INTRODUCTION AND SUMMARY

1.1. THE STATISTICS S_n AND $Q_{n,x_0}(\lambda)$. Let (X_i, Y_i) , $i=1, \dots, n$, be independent bivariate random variables (rv's) with common continuous bivariate distribution function $H(x, y)$, $-\infty < x, y < \infty$. Let $X_{(1:n)} \leq X_{(2:n)} \leq \dots \leq X_{(n:n)}$ denote the X-variate order statistics, and $Y_{[1:n]}, Y_{[2:n]}, \dots, Y_{[n:n]}$ denote the corresponding Y-variates associated, respectively, with the above ordered $X_{(i:n)}$'s. The $Y_{[i:n]}$'s have been termed in the literature as induced order statistics or concomitants of order statistics. Asymptotic distribution theory of functions of these induced order statistics, including their appropriate linear combination have been studied by several authors (e.g., Bhattacharya (1974), David, O'Connell et al. (1977) and Yang (1977) and (1981)). Their results are useful in application to non-parametric testing and estimation problems concerning the regression function $m(x) = E(Y|X=x)$ as well as to certain Y-variate prediction and selection problems based on information contained in X-variate order statistics. In Part I of this thesis, containing Chapters II and III, we study the asymptotic normality of a linear combination of induced order statistics $Y_{[i:n]}$'s, with double weights, namely, S_n defined by (1.1.1) below, where the weights depend on the ranks of $X_{(i:n)}$'s (among the X-observations) and ranks of $Y_{[i:n]}$'s (among the Y-observations):

$$(1.1.1) \quad S_n = n^{-1} \sum_{i=1}^n J\left(\frac{i}{n}, \frac{R_{in}}{n}\right) Y_{[i:n]}$$

where R_{in} is the rank of $Y_{[i:n]}$ among the Y -observations and $J(\dots)$ is a two dimensional score function satisfying certain regularity conditions (see Theorem 3.1.1 below and also [11]). These results are useful in applications to the problems of testing and estimation of parameters of conditional distribution of Y given $X = x$. Our results extend those of Yang (1981) and Stigler (1974).

We also study in Part II of this thesis, containing chapters IV and V, the asymptotic normality of a closely related statistic $Q_{n,x_0}(\lambda)$, defined by (1.1.2) below, which is a nonparametric estimator of the λ th quantile, $0 < \lambda < 1$, of the conditional distribution of Y given $X = x_0$, proposed by Mehra (1986):

$$(1.1.2) \quad Q_{n,x_0}(\lambda) = \frac{1}{na_n^2} \sum_{i=1}^n K \left(\frac{G_n(X_{[i:n]}) - G_n(x_0)}{a_n} \right) \cdot \frac{F_{n,x_0}(Y_{[i:n]})^{-\lambda}}{a_n} Y_{[i:n]}$$

$$\equiv \frac{1}{na_n^2} \sum_{i=1}^n K \left(\frac{G_n(X_i) - G_n(x_0)}{a_n} \right) \cdot \frac{F_{n,x_0}(Y_i)^{-\lambda}}{a_n} Y_i,$$

$$-\infty < x_0 < \infty, 0 < \lambda < 1$$

where $\{a_n\}$ is a sequence of positive numbers which goes to zero as $n \rightarrow \infty$,

$$(1.1.3) \quad G_n(x) = \frac{1}{n} \sum_{i=1}^n I(X_i \leq x)$$

is the marginal empirical distribution function of the X -observations,

$$(1.1.4) \quad F_{n,x_0}(y) = \frac{1}{na_n} \sum_{j=1}^n K^* \left(\frac{G_n(X_j) - G_n(x_0)}{a_n} \right) I(Y_j \leq y)$$

is the "conditional empirical distribution function" of Y -observations given $X = x_0$, and $K^*(\cdot)$ and $K(\cdot, \cdot)$ are, respectively, a univariate and a bivariate kernel functions satisfying certain regularity conditions (see (4.1.3) to (4.1.6)). $Q_{n, x_0}(\lambda)$, defined by (1.1.2), is related to the class of statistics S_n defined by (1.1.1) in the sense that it is also a linear combination of induced order statistics with double weights, but is not covered by that class since the weights in $Q_{n, x_0}(\lambda)$ depend on the ranks of $X_{(1:n)}$ (among X 's) and on the "generalized ranks" of $Y_{[1:n]}$ (among Y 's) unlike in (1.1.1) where the weights depend on the ranks of $X_{(1:n)}$'s and the ranks of $Y_{[1:n]}$'s only.

The methods of proof employed in proving our main Theorems in the two cases are similar in that both are based on the "Projection Technique" introduced by Hajek (1968).

1.2 A BRIEF SUMMARY OF THE RESULTS. In Chapter II, we approximate S_n by a sum of independent and identically distributed (iid) rv's using the "Projection Technique" and prove some preliminary results which assist us in Chapter III.

Chapter III is concerned with the asymptotic normality of S_n for which we make use of the results obtained in Chapter II, and with obtaining the results in Theorem 1 of Yang (1981) and those in Theorem 2 of Stigler (1974) as special cases.

In Chapter IV, we prove the weak consistency of $Q_{n, x_0}(\lambda)$.

Chapter V deals with the asymptotic normality of $Q_{n, x_0}(\lambda)$ utilising the results obtained in Chapter IV. The results of Chapter IV and Chapter V are

derived under certain regularity conditions on the sequence $\{a_n\}$, the kernel functions K^* and K , and the joint distribution $H(x,y)$, $-\infty < x, y < \infty$ of (X,Y) .

1.3 NOTATION AND TERMINOLOGY. Most of the notation is introduced in the text as and when necessary. However, the following terms should be noted, as they appear more frequently in the text.

(i) C_1, C_2 etc., denote absolute positive constants, not necessarily representing the same value in each appearance.

(ii) The inverse of a function is defined to be left continuous:

$$G^{-1}(t) = \inf \{x: G(x) \geq t\}.$$

(iii) $K'_1(\theta_1, \theta_2), K''_1(\theta_1, \theta_2)$ etc., (or $K_1^{(m)}(\theta_1, \theta_2), m = 1, 2, \dots$) denote respectively, the first order partial derivative, second order partial derivative etc., of the function K with respect to the first argument, evaluated at (θ_1, θ_2) . However the notation $K_1^{(m)}(\theta_1, \theta_2)$ is used in case of higher order derivatives only, for convenience. $K'_2(\theta_1, \theta_2), K''_2(\theta_1, \theta_2)$ etc., and $K_2^{(m)}(\theta_1, \theta_2)$, are similarly defined. $K_{12}^{(m_1, m_2)}(\theta_1, \theta_2)$ denote the $(m_1 + m_2)$ -th order mixed derivative of K obtained by differentiating m_1 times with respect to the first argument and m_2 times with respect to the second argument, and evaluated at (θ_1, θ_2) .

(iv) $g'(\theta), g''(\theta)$ etc., (or $g^{(m)}(\theta), m=1, 2, \dots$) denote the first derivative, second derivative etc., of g evaluated at θ .

(v) For any real numbers x and y , $x \wedge y$ indicates $\min(x, y)$.

(vi) The order symbols O , o , O_p and o_p are used in their standard notation.

(vii) $I(A)$ denotes the indicator function of the set A , and $\square$ signals the end of a proof. The following abbreviations are also used:

$$\sum_1^n \equiv \sum_{i=1}^n \text{ unless, otherwise specified,}$$

$$\int = \int_{-\infty}^{\infty} \text{ unless, otherwise specified.}$$

CHAPTER II

PROJECTION $\hat{S}_n$ OF S_n AND VARIANCE OF $\hat{S}_n$

The object of Chapters II and III is to establish the asymptotic normality of statistics of the form S_n (see 2.1.1. below) which is a linear combination of induced order statistics with double weights, weights depending on the ranks of the observations. In the present Chapter namely, Chapter II, we shall prove some preliminary results concerning S_n which assist us in proving the asymptotic normality of S_n (in a properly normalized form) in Chapter III. A brief outline of the method of proof employed in proving the asymptotic normality of S_n , namely, Hajek's projection approach is given in Section 2.1. In Section 2.2 we find the projection $\hat{S}_n$ of S_n and the asymptotic variance of $\hat{S}_n$ is obtained in Section 2.3.

2.1 PRELIMINARIES. Let $(X_1, Y_1), \dots, (X_n, Y_n)$ be iid rv's, with common continuous bivariate distribution function $H(x, y)$, $-\infty < x, y < \infty$. Let $X_{(1:n)} \leq X_{(2:n)} \leq \dots \leq X_{(n:n)}$ denote the X-variate order-statistics and $Y_{[1:n]}, Y_{[2:n]}, \dots, Y_{[n:n]}$ denote the corresponding Y-variate induced order statistics. We consider statistics of the form.

$$(2.1.1) \quad S_n = \frac{1}{n} \sum_{i=1}^n J\left(\frac{i}{n}, \frac{R_{in}}{n}\right) Y_{[i:n]}$$

where R_{in} is the rank of $Y_{[i:n]}$ when $Y_{[1:n]}$'s, $1 \leq i \leq n$, are arranged in ascending order of the values and $J(u, v)$, $0 \leq u, v \leq 1$ is a suitable bounded score function satisfying certain regularity conditions.

Our method of proof in proving the asymptotic normality of S_n is the same as that employed by Stigler [17] and Yang [22], namely, the use of the Projection technique introduced originally by Hajek. This method enables us to demonstrate that S_n can be suitably approximated in mean square by its projection $\hat{S}_n$ (defined below), as $n \rightarrow \infty$. $\hat{S}_n$ being a sum of independent r.v.'s with finite second moments, the above approximation and the standard central limit theorem gives the asymptotic normality of S_n . For convenience, we state Hajek's [7] projection lemma here as lemma 2.1.1.

Lemma 2.1.1. (Hajek's (1968) Projection Lemma). Let $Z_1, Z_2, \dots, Z_n$ be independent r.v.'s and F be the Hilbert space of almost sure equivalence classes of square integrable statistics depending on $Z_1, \dots, Z_n$. Let S be the closed linear subspace of F consisting of statistics of the form

$$L = \sum_{i=1}^n \ell_i(Z_i), \text{ where } \ell_i \text{ are functions such that } E[\ell_i^2(Z_i)] < \infty. \text{ Then, if}$$

$S \in F$, the projection of S on S is given by

$$(2.1.2) \quad \hat{S} = \sum_{i=1}^n E(S|Z_i) - (n-1)E(S).$$

Thus $E(\hat{S}) = E(S)$ and

$$(2.1.3) \quad E(S - \hat{S})^2 = \sigma^2(S) - \sigma^2(\hat{S}).$$

2.2 PROJECTION $\hat{S}_n$ OF S_n . Following Lemma 2.1.1 we define the projection $\hat{S}_n$ of S_n as

$$(2.2.1) \quad \hat{S}_n = \sum_{k=1}^n E(S_n | X_k, Y_k) - (n-1)E(S_n).$$

Lemma 2.2.1 The projection $\hat{S}_n$ of S_n (as defined by (2.2.1) is given by

$$\begin{aligned}
 (2.2.2) \quad \hat{S}_n = & n^{-1} \sum_{k=1}^n Y_k \left[\sum_{i=1}^n \sum_{s=1}^n J\left(\frac{i}{n}, \frac{s}{n}\right) p_{i-1, s-1}^{(n-1)}(X_k, Y_k) \right] \\
 & + n^{-1} \sum_{k=1}^n \left[(n-1) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{i=1}^{n-1} \sum_{s=1}^{n-1} J\left(\frac{i+1}{n}, \frac{s}{n}\right) p_{i-1, s-1}^{(n-2)}(u, v) \right. \\
 & \quad \times I(X_k \leq u, Y_k > v) v dH(u, v) \\
 & + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\sum_{i=1}^{n-1} \sum_{s=1}^{n-1} J\left(\frac{i+1}{n}, \frac{s+1}{n}\right) p_{i-1, s-1}^{(n-2)}(u, v) \right] I(X_k \leq u, Y_k \leq v) v dH(u, v) \\
 & + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\sum_{i=1}^{n-1} \sum_{s=1}^{n-1} J\left(\frac{i}{n}, \frac{s}{n}\right) p_{i-1, s-1}^{(n-2)}(u, v) \right] I(X_k > u, Y_k > v) v dH(u, v) \\
 & \left. + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\sum_{i=1}^{n-1} \sum_{s=1}^{n-1} J\left(\frac{i}{n}, \frac{s+1}{n}\right) p_{i-1, s-1}^{(n-2)}(u, v) \right] I(X_k > u, Y_k \leq v) v dH(u, v) \right] \\
 = & \eta_n,
 \end{aligned}$$

where $\eta_n = (n-1)E(S_n)$, is a constant depending on n, J and H , and

$$\begin{aligned}
 (2.2.3) \quad p_{i,s}^{(n)}(u, v) = & P[\text{exactly } i \text{ X's are } < u \text{ and exactly } s \text{ Y's are } < v \\
 & \text{out of } n \text{ independently and identically distributed} \\
 & \text{r.v.'s } (X_k, Y_k), k = 1, 2, \dots, n]
 \end{aligned}$$

$$= \sum_{k=0}^{\min(i,s)} \frac{n!}{k!(i-k)!(s-k)!(n-i-s+k)!} \theta_1^k \theta_2^{i-k} \theta_3^{s-k} \theta_4^{n-i-s+k}$$

$$\begin{aligned}
 \text{with } \theta_1 = & P(X_1 \leq u, Y_1 \leq v), & \theta_2 = & P(X_1 \leq u, Y_1 > v), \\
 \theta_3 = & P(X_1 > u, Y_1 \leq v) \quad \text{and} & \theta_4 = & P(X_1 > u, Y_1 > v),
 \end{aligned}$$

for $i, s = 0, 1, 2, \dots, n$.

PROOF. From (2.1.2) we have

$$(2.2.4) \quad S_n = n^{-1} \sum_{i=1}^n \sum_{k=1}^n E\left[J\left(\frac{i}{n}, \frac{R_{in}}{n}\right) Y_{[i:n]} | X_k, Y_k\right] - (n-1)E(S_n) \\ = n^{-1} \sum_{k=1}^n \sum_{i=1}^n E\left[J\left(\frac{i}{n}, \frac{R_{in}}{n}\right) Y_{[i:n]} | X_k, Y_k\right] - (n-1)E(S_n).$$

We shall first evaluate the conditional expectations involved in (2.2.4). For $1 < i < n$, we may write

$$(2.2.5) \quad E\left[J\left(\frac{i}{n}, \frac{R_{in}}{n}\right) Y_{[i:n]} | X_n = x_n, Y_n = y_n\right] \\ = E\left[J\left(\frac{i}{n}, \frac{R_{in}}{n}\right) Y_{[i:n]} I(X_{(i-1:n-1)} \leq x_n \leq X_{(i:n-1)}) | X_n = x_n, Y_n = y_n\right] \\ + E\left[J\left(\frac{i}{n}, \frac{R_{in}}{n}\right) Y_{[i:n]} I(x_n < X_{(i-1:n-1)}) | X_n = x_n, Y_n = y_n\right] \\ + E\left[J\left(\frac{i}{n}, \frac{R_{in}}{n}\right) Y_{[i:n]} I(X_n > X_{(i:n-1)}) | X_n = x_n, Y_n = y_n\right] \\ = \textcircled{A} + \textcircled{B} + \textcircled{C}.$$

Since $R_{in} = \sum_{j=1}^n I(Y_j \leq Y_{[i:n]})$, using the obvious implications

$$X_{(i-1:n-1)} \leq x_n \leq X_{(i:n-1)} \Rightarrow X_{(i:n)} = x_n \Rightarrow Y_{[i:n]} = y_n,$$

$$\textcircled{A} = y_n E\left[J\left(\frac{i}{n}, \frac{\sum_{j=1}^n I(Y_j \leq y_n)}{n}\right) I(X_{(i-1:n-1)} \leq x_n \leq X_{(i:n-1)}) | X_n = x_n, Y_n = y_n\right]$$

continued

$$\begin{aligned}
&= y_n E \left[J \left(\frac{1}{n}, \frac{1 + \sum_{j=1}^{n-1} I(Y_j \leq y_n)}{n} \right) I(X_{(i-1:n-1)} \leq x_n \leq X_{(i:n-1)}) \right] \\
&= y_n \sum_{s=0}^{n-1} J \left(\frac{1}{n}, \frac{s+1}{n} \right) p_{i-1,s}^{(n-1)}(x_n, y_n) \\
&= y_n \sum_{s=1}^n J \left(\frac{1}{n}, \frac{s}{n} \right) p_{i-1,s-1}^{(n-1)}(x_n, y_n); \\
\textcircled{B} &= E \left[J \left(\frac{1}{n}, \frac{\sum_{j=1}^n I(Y_j \leq Y_{[i:n]})}{n} \right) Y_{[i:n]} I(X_n < X_{(i-1:n-1)}) | X_n = x_n, Y_n = y_n \right] \\
&= E \left[J \left(\frac{1}{n}, \frac{I(y_n < Y_{[i:n]}) + \sum_{j=1}^{n-1} I(Y_j \leq Y_{[i:n]})}{n} \right) Y_{[i:n]} I(X_n < X_{(i-1:n-1)}) | X_n = x_n, Y_n = y_n \right]
\end{aligned}$$

from which, using the implication

$$X_n < X_{(i-1:n-1)} \Rightarrow X_{(i:n)} = X_{(i-1:n-1)} \Rightarrow \bar{Y}_{[i:n]} = Y_{[i-1:n-1]},$$

we obtain

$$\begin{aligned}
\textcircled{B} &= E \left[J \left(\frac{1}{n}, \frac{I(y_n < Y_{[i-1:n-1]}) + \sum_{j=1}^{n-1} I(Y_j \leq Y_{[i-1:n-1]})}{n} \right) Y_{[i-1:n-1]} I(X_n < X_{(i-1:n-1)}) \right] \\
&= (n-1) \iint v E \left[J \left(\frac{1}{n}, \frac{I(y_n < v) + \sum_{j=1}^{n-1} I(Y_j \leq v)}{n} \right) I \left(\sum_{j=1}^{n-1} I(X_j < u) = i-2 \right) I(X_n < u) dH(u, v) \right] \\
&= (n-1) \iint v \left[\sum_{s=1}^{n-1} J \left(\frac{1}{n}, \frac{s}{n} \right) p_{i-2,s-1}^{(n-2)}(u, v) I(y_n > v) I(X_n < u) dH(u, v) \right. \\
&\quad \left. + \sum_{s=1}^{n-1} J \left(\frac{1}{n}, \frac{s+1}{n} \right) p_{i-2,s-1}^{(n-2)}(u, v) I(y_n \leq v) I(X_n < u) dH(u, v) \right] \\
&= (n-1) \iint \left[\sum_{s=1}^{n-1} J \left(\frac{1}{n}, \frac{s}{n} \right) p_{i-2,s-1}^{(n-2)}(u, v) I(X_n < u, y_n > v) v dH(u, v) \right.
\end{aligned}$$

continued

$$+ (n-1) \iint \left[\sum_{s=1}^{n-1} J\left(\frac{1}{n}, \frac{s+1}{n}\right) p_{i-2,s-1}^{(n-2)}(u,v) \right] I(x_n \leq u, y_n \leq v) v dH(u,v), \text{ and}$$

proceeding on the same lines as for B, we find that

$$\begin{aligned} \textcircled{C} &= (n-1) \iint \left[\sum_{s=1}^{n-1} J\left(\frac{1}{n}, \frac{s}{n}\right) p_{i-1,s-1}^{(n-2)}(u,v) \right] I(x_n > u, y_n > v) v dH(u,v) \\ &+ (n-1) \iint \left[\sum_{s=1}^{n-1} J\left(\frac{1}{n}, \frac{s+1}{n}\right) p_{i-1,s-1}^{(n-2)}(u,v) \right] I(x_n > u, y_n \leq v) v dH(u,v). \end{aligned}$$

Hence for $1 < i < n$, we have from (2.2.5) and the expressions for $\textcircled{A}$,

$\textcircled{B}$ and $\textcircled{C}$,

$$\begin{aligned} (2.2.6) \quad & E\left[J\left(\frac{1}{n}, \frac{R_{1n}}{n}\right) Y_{[1:n]} | X_n, Y_n\right] \\ &= Y_n \sum_{s=1}^n J\left(\frac{1}{n}, \frac{s}{n}\right) p_{i-1,s-1}^{(n-1)}(X_n, Y_n) \\ &+ (n-1) \iint \left[\sum_{s=1}^{n-1} J\left(\frac{1}{n}, \frac{s}{n}\right) p_{i-2,s-1}^{(n-2)}(u,v) \right] I(X_n \leq u, Y_n > v) v dH(u,v) \\ &+ (n-1) \iint \left[\sum_{s=1}^{n-1} J\left(\frac{1}{n}, \frac{s+1}{n}\right) p_{i-2,s-1}^{(n-2)}(u,v) \right] I(X_n \leq u, Y_n \leq v) v dH(u,v) \\ &+ (n-1) \iint \left[\sum_{s=1}^{n-1} J\left(\frac{1}{n}, \frac{s}{n}\right) p_{i-1,s-1}^{(n-2)}(u,v) \right] I(X_n > u, Y_n > v) v dH(u,v) \\ &+ (n-1) \iint \left[\sum_{s=1}^{n-1} J\left(\frac{1}{n}, \frac{s+1}{n}\right) p_{i-1,s-1}^{(n-2)}(u,v) \right] I(X_n > u, Y_n \leq v) v dH(u,v). \end{aligned}$$

Now, for $i = 1$, we may write

$$E\left[J\left(\frac{1}{n}, \frac{R_{1n}}{n}\right) Y_{[1:n]} | X_n = x_n, Y_n = y_n\right]$$

continued

$$\begin{aligned}
&= E\left[J\left(\frac{1}{n}, \frac{R_{1n}}{n}\right)Y_{[1:n]}I(X_n \leq X_{(1:n-1)}) \mid X_n = x_n, Y_n = y_n\right] \\
&\quad + E\left[J\left(\frac{1}{n}, \frac{R_{1n}}{n}\right)Y_{[1:n]}I(X_n > X_{(1:n-1)}) \mid X_n = x_n, Y_n = y_n\right] \\
&= \textcircled{D} + \textcircled{E}.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
\textcircled{D} &= y_n \sum_{s=1}^n J\left(\frac{1}{n}, \frac{s}{n}\right) p_{0,s-1}^{(n-1)}(x_n, y_n) \quad \text{and} \\
\textcircled{E} &= (n-1) \iint \left[\sum_{s=1}^{n-1} J\left(\frac{1}{n}, \frac{s}{n}\right) p_{0,s-1}^{(n-2)}(u, v) \right] I(x_n > u, y_n > v) v dH(u, v) \\
&\quad + (n-1) \iint \left[\sum_{s=1}^{n-1} J\left(\frac{1}{n}, \frac{s+1}{n}\right) p_{0,s-1}^{(n-2)}(u, v) \right] I(x_n > u, y_n \leq v) v dH(u, v),
\end{aligned}$$

so that,

$$\begin{aligned}
(2.2.7) \quad E\left[J\left(\frac{1}{n}, \frac{R_{1n}}{n}\right)Y_{[1:n]} \mid X_n, Y_n\right] \\
&= y_n \sum_{s=1}^n J\left(\frac{1}{n}, \frac{s}{n}\right) p_{0,s-1}^{(n-1)}(x_n, y_n) \\
&\quad + (n-1) \iint \left[\sum_{s=1}^{n-1} J\left(\frac{1}{n}, \frac{s}{n}\right) p_{0,s-1}^{(n-2)}(u, v) \right] I(x_n > u, y_n > v) v dH(u, v) \\
&\quad + (n-1) \iint \left[\sum_{s=1}^{n-1} J\left(\frac{1}{n}, \frac{s+1}{n}\right) p_{0,s-1}^{(n-2)}(u, v) \right] I(x_n > u, y_n \leq v) v dH(u, v).
\end{aligned}$$

For $i = n$, we may write

$$\begin{aligned}
&E\left[J\left(1, \frac{R_{nn}}{n}\right)Y_{[n:n]} \mid X_n, Y_n\right] \\
&= E\left[J\left(1, \frac{R_{nn}}{n}\right)Y_{[n:n]}I(X_n > X_{(n-1:n-1)}) \mid X_n, Y_n\right] \\
&\quad + E\left[J\left(1, \frac{R_{nn}}{n}\right)Y_{[n:n]}I(X_n \leq X_{(n-1:n-1)}) \mid X_n, Y_n\right].
\end{aligned}$$

Again, it is easy to see that

$$\begin{aligned}
(2.2.8) \quad & E\left[J\left(1, \frac{R_{nn}}{n}\right) Y_{[n:n]} | X_n, Y_n\right] \\
&= Y_n \sum_{s=1}^n J\left(1, \frac{s}{n}\right) p_{n-1,s-1}^{(n-1)}(X_n, Y_n) \\
&\quad + (n-1) \iint \left[\sum_{s=1}^{n-1} J\left(1, \frac{s}{n}\right) p_{n-2,s-1}^{(n-2)}(u,v) \right] I(X_n < u, Y_n > v) v dH(u,v) \\
&\quad + (n-1) \iint \left[\sum_{s=1}^{n-1} J\left(1, \frac{s+1}{n}\right) p_{n-2,s-1}^{(n-2)}(u,v) \right] I(X_n < u, Y_n < v) v dH(u,v).
\end{aligned}$$

A symmetric argument shows that when conditioned on any (X_k, Y_k) , the expressions for the conditional expectations in (2.2.6), (2.2.7) and (2.2.8) remain valid with (X_n, Y_n) replaced by (X_k, Y_k) for $k = 1, 2, \dots, n-1$.

Now (2.2.2) follows by using the above expressions (2.2.6) to (2.2.8) in (2.2.4) and this proves the lemma. $\square$

The following lemma is useful in section 2.3.

Lemma 2.2.2. Let $K(\dots)$ be any bounded continuous function. Let (X_i, Y_i) , $i = 1, \dots, n$ be independently identically distributed with a continuous d.f. $H(x, y)$, $-\infty < x, y < \infty$ with marginal d.f.'s of X and Y denoted by $G(x)$ and $F(y)$, respectively. Also for any fixed real numbers u and v let $p_{i,s}^{(n)}(u, v)$ be as defined by (2.2.3) above.

Then, for fixed $-\infty < u, v < \infty$,

$$(i) \quad \sum_{i=0}^n \sum_{s=0}^n K\left(\frac{i}{n}, \frac{s}{n}\right) p_{i,s}^{(n)}(u, v) \rightarrow K(G(u), F(v)) \text{ as } n \rightarrow \infty.$$

$$(ii) \quad \sum_{i=0}^n \sum_{s=0}^n K\left(a_n \left(\frac{i}{n}\right) + b_n, c_n \left(\frac{s}{n}\right) + d_n\right) p_{i,s}^{(n)}(u, v) \rightarrow K(G(u), F(v)),$$

where sequences of constants $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ satisfy

$$a_n \rightarrow 1, \quad c_n \rightarrow 1, \quad b_n \rightarrow 0 \quad \text{and} \quad d_n \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

PROOF: First note that the r.v.'s $T_1 = \{\# X's < u\}$ and $T_2 = \{\# Y's < v\}$ are Binomially distributed with parameters $(n, G(u))$ and $(n, F(v))$ respectively, and that

$$p_{i,s}^{(n)}(u,v) = P(T_1 = i, T_2 = s).$$

Let $B_n = \{(i,s): |\frac{i}{n} - G(u)| \leq n^{-1/3}, |\frac{s}{n} - F(v)| \leq n^{-1/3}\}.$

Write

$$\begin{aligned} (2.2.9) \quad & \left| \sum_{i=0}^n \sum_{s=0}^n K\left(\frac{i}{n}, \frac{s}{n}\right) p_{i,s}^{(n)}(u,v) - K(G(u), F(v)) \right| \\ & \leq \sum_{i=0}^n \sum_{s=0}^n \left| K\left(\frac{i}{n}, \frac{s}{n}\right) - K(G(u), F(v)) \right| p_{i,s}^{(n)}(u,v) \\ & = \sum_{(i,s) \in B_n} \left| K\left(\frac{i}{n}, \frac{s}{n}\right) - K(G(u), F(v)) \right| p_{i,s}^{(n)}(u,v) \\ & \quad + \sum_{(i,s) \in B_n^c} \left| K\left(\frac{i}{n}, \frac{s}{n}\right) - K(G(u), F(v)) \right| p_{i,s}^{(n)}(u,v) \end{aligned}$$

Now

$$\begin{aligned} \sum_{(i,s) \in B_n} \left| K\left(\frac{i}{n}, \frac{s}{n}\right) - K(G(u), F(v)) \right| p_{i,s}^{(n)}(u,v) & \leq w(K; n^{-1/3}, n^{-1/3}) \sum_{(i,s) \in B_n} p_{i,s}^{(n)}(u,v) \\ & \leq w(K; n^{-1/3}, n^{-1/3}) \end{aligned}$$

where

$$\begin{aligned} (2.2.10) \quad & w(K; u, v) = \text{modulus of continuity of } K \text{ at } (u, v) \\ & = \sup_{\substack{|x_1 - x_2| \leq u \\ |y_1 - y_2| \leq v}} |K(x_1, y_1) - K(x_2, y_2)| \end{aligned}$$

But $w(K; n^{-\alpha}, n^{-\alpha}) \rightarrow 0$ as $n \rightarrow \infty$ for $\alpha > 0$, since K is a continuous function. Hence the first term on the R.H.S. of (2.2.9) $\rightarrow 0$ as $n \rightarrow \infty$.

$$\begin{aligned} \sum_{(i,s) \in B_n^c} \left| K\left(\frac{i}{n}, \frac{s}{n}\right) - K(G(u), F(v)) \right| p_{i,s}^{(n)}(u,v) &\leq 2 \sup_{x,y} |K(x,y)| \sum_{(i,s) \in B_n^c} p_{i,s}^{(n)}(u,v) \\ &= 2 \sup_{x,y} |K(x,y)| P\left(\left|\frac{T_1}{n} - G(u)\right| \leq n^{-1/3}, \left|\frac{T_2}{n} - F(v)\right| \leq n^{-1/3}\right)^c \\ &\leq 2 \sup_{x,y} |K(x,y)| [P\left(\left|\frac{T_1}{n} - G(u)\right| > n^{-1/3}\right) + P\left(\left|\frac{T_2}{n} - F(v)\right| > n^{-1/3}\right)] \end{aligned}$$

Applying Tchebychev's inequality to the binomial probabilities on the R.H.S. and since K is assumed bounded we find that the R.H.S. $\rightarrow 0$ as $n \rightarrow \infty$. Thus both terms on the R.H.S. of (2.2.9) go to 0 as $n \rightarrow \infty$ and hence (i) of the lemma follows.

(ii) We write

$$\begin{aligned} (2.2.11) \quad & \left| \sum_{i=0}^n \sum_{s=0}^n K\left(a_n \cdot \frac{i}{n} + b_n, c_n \cdot \frac{s}{n} + d_n\right) p_{i,s}^{(n)}(u,v) - K(G(u), F(v)) \right| \\ & \leq \sum_{i=0}^n \sum_{s=0}^n \left| K\left(a_n \cdot \frac{i}{n} + b_n, c_n \cdot \frac{s}{n} + d_n\right) - K\left(\frac{i}{n}, \frac{s}{n}\right) \right| p_{i,s}^{(n)}(u,v) \\ & \quad + \left| \sum_{i=0}^n \sum_{s=0}^n K\left(\frac{i}{n}, \frac{s}{n}\right) p_{i,s}^{(n)}(u,v) - K(G(u), F(v)) \right| \\ & = (1) + (2), \text{ say.} \end{aligned}$$

(2) goes to zero as $n \rightarrow \infty$ by part (i) of the lemma.

Now

$$(1) \leq \sum_{i=0}^n \sum_{s=0}^n w(K; |(a_n-1)\frac{i}{n} + b_n|, |(c_n-1)\frac{s}{n} + d_n|) p_{i,s}^{(n)}(u,v)$$

where w is the modulus of continuity as defined by (2.2.10). Since $W(K;u,v)$ is nondecreasing in both u and v for any continuous function K , we can write

$$(1) \leq \sum_{i=0}^n \sum_{s=0}^n W(K; |a_n-1| + |b_n|, |c_n-1| + |d_n|) p_{i,s}^{(n)}(u,v) \\ = W(K; |a_n-1| + |b_n|, |c_n-1| + |d_n|)$$

which goes to zero as $n \rightarrow \infty$ since $a_n \rightarrow 1$, $c_n \rightarrow 1$, $b_n \rightarrow 0$, $d_n \rightarrow 0$ as $n \rightarrow \infty$ and K is a continuous function. Hence (ii) of the lemma follows from (2.2.11). $\square$

2.3 ASYMPTOTIC VARIANCE OF $\hat{S}_n$. We shall demonstrate in Chapter III that the statistics $T_n = n^{1/2}(S_n - ES_n)$ and $\hat{T}_n = n^{1/2}(\hat{S}_n - E\hat{S}_n)$ are asymptotically equivalent in mean square by showing that $E(T_n - \hat{T}_n)^2 = \text{Var}(T_n) - \text{Var}(\hat{T}_n) \rightarrow 0$ as $n \rightarrow \infty$. As a first step towards that, in this section we find the asymptotic variance of $\hat{T}_n$.

Let $Q_n, \bar{Q}_n, Q_n^*, \bar{Q}_n^*$ and $\bar{Q}_n$ denote the following expressions:

$$(2.3.1) \quad Q_n(u,v) = \sum_{i=1}^{n-1} \sum_{s=1}^{n-1} J\left(\frac{i+1}{n}, \frac{s}{n}\right) p_{i-1,s-1}^{(n-2)}(u,v) \\ Q_n^*(u,v) = \sum_{i=1}^{n-1} \sum_{s=1}^{n-1} J\left(\frac{1}{n}, \frac{s}{n}\right) p_{i-1,s-1}^{(n-2)}(u,v) \\ \bar{Q}_n(u,v) = \sum_{i=1}^n \sum_{s=1}^n J\left(\frac{1}{n}, \frac{s}{n}\right) p_{i-1,s-1}^{(n-1)}(u,v)$$

$$(2.3.1) \quad \bar{Q}_n(u, v) = \sum_{i=1}^{n-1} \sum_{s=1}^{n-1} J\left(\frac{i+1}{n}, \frac{s+1}{n}\right) p_{i-1, s-1}^{(n-2)}(u, v)$$

$$\bar{Q}_n^*(u, v) = \sum_{i=1}^{n-1} \sum_{s=1}^{n-1} J\left(\frac{i}{n}, \frac{s}{n}\right) p_{i-1, s-1}^{(n-2)}(u, v).$$

Using the above notation and the fact that

$$I(X_k \leq u, Y_k > v) = I(X_k \leq u) - I(X_k \leq u, Y_k \leq v)$$

$$I(X_k > u, Y_k > v) = 1 - I(X_k \leq u) - I(X_k > u, Y_k \leq v)$$

$$I(X_k > u, Y_k \leq v) = I(Y_k \leq v) - I(X_k \leq u, Y_k \leq v),$$

(2.2.2) may be rewritten as

$$(2.3.2) \quad \hat{S}_n = n^{-1} \sum_{k=1}^n Y_k \bar{Q}_n(X_k, Y_k)$$

$$+ n^{-1} \sum_{k=1}^n (n-1) [\iint [Q_n(u, v) - Q_n^*(u, v)] I(X_k \leq u) v dH(u, v)$$

$$+ \iint [\bar{Q}_n(u, v) - Q_n(u, v) + Q_n^*(u, v) - \bar{Q}_n^*(u, v)] I(X_k \leq u, Y_k \leq v) v dH(u, v)$$

$$+ \iint [\bar{Q}_n^*(u, v) - Q_n^*(u, v)] I(Y_k \leq v) v dH(u, v)$$

$$+ \iint Q_n^*(u, v) v dH(u, v)] - \eta_n.$$

We make the following assumptions on J :

$$(2.3.3) \quad J(u, v), 0 \leq u, v \leq 1 \text{ is a continuous function.}$$

(2.3.4) $J(.,.)$ possesses bounded continuous partial derivatives up to the third order.

LEMMA 2.3.1. Suppose that J satisfies assumptions (2.3.3) and (2.3.4). Then for any real numbers u,v such that $0 < G(u), F(v) < 1$ the following assertions hold:

- (a) $\bar{Q}_n(u,v) \rightarrow J(G(u), F(v))$ as $n \rightarrow \infty$.
- (2.3.5) (b) $(n-1)[\bar{Q}_n(u,v) - Q_n(u,v)] \rightarrow J'_2(G(u), F(v))$ as $n \rightarrow \infty$.
- (c) $(n-1)[\bar{Q}_n^*(u,v) - Q_n^*(u,v)] \rightarrow J'_2(G(u), F(v))$ as $n \rightarrow \infty$.
- (d) $(n-1)[Q_n(u,v) - Q_n^*(u,v)] \rightarrow J'_1(G(u), F(v))$ as $n \rightarrow \infty$.

(In (2.3.5) above $J'_1(\dots)$ and $J'_2(\dots)$ denote, respectively, the first order partial derivatives w.r. to the first and second arguments. The notation $J''_1(\dots)$, $J''_2(\dots)$ and $J''_{12}(\dots)$ used below have a similar obvious connotation.)

PROOF:

$$\begin{aligned} \text{(a) Write } \bar{Q}_n(u,v) &= \sum_{i=1}^n \sum_{s=1}^n J\left(\frac{i}{n}, \frac{s}{n}\right) p_{i-1,s-1}^{(n-1)}(u,v) \\ &= \sum_{i=0}^{n-1} \sum_{s=0}^{n-1} J\left(\frac{i+1}{n}, \frac{s+1}{n}\right) p_{i,s}^{(n-1)}(u,v). \end{aligned}$$

The result follows by using (ii) of lemma 2.2.2.

$$\text{(b) } \bar{Q}_n(u,v) - Q_n(u,v) = \sum_{i=1}^{n-1} \sum_{s=1}^{n-1} \left[J\left(\frac{i+1}{n}, \frac{s+1}{n}\right) - J\left(\frac{i+1}{n}, \frac{s}{n}\right) \right] p_{i-1,s-1}^{(n-2)}(u,v).$$

By Taylor's expansion of $J\left(\frac{i+1}{n}, \frac{s+1}{n}\right)$ around the point $\left(\frac{i+1}{n}, \frac{s}{n}\right)$ and

since the second derivatives are assumed bounded, we have

$$\begin{aligned}\bar{Q}_n(u,v) - Q_n(u,v) &= \frac{1}{n} \sum_{i=1}^{n-1} \sum_{s=1}^{n-1} J_2' \left(\frac{i+1}{n}, \frac{s}{n} \right) p_{i-1,s-1}^{(n-2)}(u,v) + o\left(\frac{1}{n^2}\right) \\ &= \frac{1}{n} \sum_{i=0}^{n-2} \sum_{s=0}^{n-2} J_2' \left(\frac{i+2}{n}, \frac{s+1}{n} \right) p_{i,s}^{(n-2)}(u,v) + o\left(\frac{1}{n^2}\right),\end{aligned}$$

so that

$$(2.3.6) \quad (n-1)[\bar{Q}_n(u,v) - Q_n(u,v)] = \frac{n-1}{n} \sum_{i=0}^{n-2} \sum_{s=0}^{n-2} J_2' \left(\frac{i+2}{n}, \frac{s+1}{n} \right) p_{i,s}^{(n-2)}(u,v) + o\left(\frac{1}{n}\right).$$

Since J_2' is assumed bounded and continuous, lemma (2.2.2) applies to the first term on the R.H.S. of (2.3.6) and we have

$$(n-1)[\bar{Q}_n(u,v) - Q_n(u,v)] \rightarrow J_2'(G(u), F(v)), \text{ as } n \rightarrow \infty.$$

(c) follows exactly on the same lines as (b).

(d) From (2.3.1) we have

$$Q_n(u,v) - Q_n^*(u,v) = \sum_{i=1}^{n-1} \sum_{s=1}^{n-1} \left[J \left(\frac{i+1}{n}, \frac{s}{n} \right) - J \left(\frac{i}{n}, \frac{s}{n} \right) \right] p_{i-1,s-1}^{(n-2)}(u,v)$$

Since the second derivatives of J are assumed bounded, the Taylor expansion of

$J \left(\frac{i+1}{n}, \frac{s}{n} \right)$ around $\left(\frac{i}{n}, \frac{s}{n} \right)$ yields

$$Q_n(u,v) - Q_n^*(u,v) = \frac{1}{n} \sum_{i=1}^{n-1} \sum_{s=1}^{n-1} J_1' \left(\frac{i}{n}, \frac{s}{n} \right) p_{i-1,s-1}^{(n-2)}(u,v) + o\left(\frac{1}{n^2}\right),$$

so that

$$(2.3.7) \quad (n-1)[Q_n(u,v) - Q_n^*(u,v)] = \frac{n-1}{n} \sum_{i=1}^{n-1} \sum_{s=1}^{n-1} J_1' \left(\frac{i}{n}, \frac{s}{n} \right) p_{i-1,s-1}^{(n-2)}(u,v) + o\left(\frac{1}{n}\right).$$

Since J_1' is bounded and continuous, lemma 2.2.2 applies to the first term on the R.H.S. of (2.3.7) again as in the case of (2.3.6) and we have

$$(n-1)[Q_n(u,v) - Q_n^*(u,v)] \rightarrow J_1'(G(u), F(v)), \text{ as } n \rightarrow \infty. \quad \square$$

LEMMA 2.3.2. Under the assumptions of lemma 2.3.1 we have

$$(2.3.8) \quad \lim_{n \rightarrow \infty} \text{Var}(\hat{T}_n) = \sigma^2(J, H)$$

where

$$(2.3.9) \quad \begin{aligned} \sigma^2(J, H) = & \iint J^2(G(u), F(v)) v^2 dH(u, v) \\ & + \iiint [J_1'(G(u_1), F(v_1)) J_1'(G(u_2), F(v_2)) [G(u_1 \wedge u_2) - G(u_1)G(u_2)] \\ & + J_2'(G(u_1), F(v_1)) J_2'(G(u_2), F(v_2)) [F(v_1 \wedge v_2) - F(v_1)F(v_2)] \\ & + 2J(G(u_1), F(v_1)) J_1'(G(u_2), F(v_2)) [I(u_1 \leq u_2) - G(u_2)] \\ & + 2J(G(u_1), F(v_1)) J_2'(G(u_2), F(v_2)) [I(v_1 \leq v_2) - F(v_2)] \\ & + 2J_1'(G(u_1), F(v_1)) J_2'(G(u_2), F(v_2)) [H(u_1, v_2) - G(u_1)F(v_2)] \\ & - J(G(u_1), F(v_1)) J(G(u_2), F(v_2))] v_1 v_2 dH(u_1, v_1) dH(u_2, v_2). \end{aligned}$$

PROOF: Since (X_k, Y_k) , $k = 1, 2, \dots, n$ are all independent, from equation (2.3.2), we have

$$(2.3.10) \quad \begin{aligned} \text{Var}(\hat{T}_n) &= n \text{Var}(\hat{S}_n) \\ &= \text{Var}(A_k + B_k + C_k + D_k) \end{aligned}$$

where

$$(2.3.11) \quad \begin{aligned} A_k &= Y_k \bar{Q}_n(X_k, Y_k) \\ B_k &= \iint (n-1) [Q_n(u, v) - Q_n^*(u, v)] I(X_k \leq u) v dH(u, v) \end{aligned}$$

$$C_k = \iint (n-1) [\bar{Q}_n(u,v) - Q_n(u,v) + Q_n^*(u,v) - \bar{Q}_n^*(u,v)] I(X_{k-} \leq u, Y_{k-} \leq v) dH(u,v)$$

$$D_k = \iint (n-1) [\bar{Q}_n^*(u,v) - Q_n^*(u,v)] I(Y_{k-} \leq v) dH(u,v).$$

We shall first obtain the expressions in the limit, for each of the variance and covariance terms on the R.H.S. and then put them all together to obtain the final limiting expression for $\text{Var}(T_n)$

$$\begin{aligned} \text{Var}(A_k) &= \iint v^2 [\bar{Q}_n(u,v)]^2 dH(u,v) - \left(\iint v \bar{Q}_n(u,v) dH(u,v) \right)^2 \\ &= \iint v^2 [\bar{Q}_n(u,v)]^2 dH(u,v) \\ &\quad - \iiint \bar{Q}_n(u_1, v_1) \bar{Q}_n(u_2, v_2) v_1 v_2 dH(u_1, v_1) dH(u_2, v_2). \end{aligned}$$

Since $|\bar{Q}_n(u,v)| \leq \sup_{x,y} |J(x,y)| < \infty$, and $EY^2 < \infty$, an application of

Lebesgue Dominated Convergence Theorem (LDCT) along with Lemma 2.3.1 gives

$$\begin{aligned} (2.3.12) \quad \lim_{n \rightarrow \infty} \text{Var}(A_k) &= \iint J^2(G(u), F(v)) v^2 dH(u,v) - \iiint J(G(u_1), F(v_1)) J(G(u_2), F(v_2)) \\ &\quad \times v_1 v_2 dH(u_1, v_1) dH(u_2, v_2). \end{aligned}$$

Now on using Fubini's theorem to take the expectation inside the integrals we see that

$$\begin{aligned} (2.3.13) \quad \text{Var}(B_k) &= \iiint (n-1) [Q_n(u_1, v_1) - Q_n^*(u_1, v_1)] (n-1) [Q_n(u_2, v_2) - Q_n^*(u_2, v_2)] \\ &\quad [G(u_1, u_2) - G(u_1)G(u_2)] v_1 v_2 dH(u_1, v_1) dH(u_2, v_2). \end{aligned}$$

By assumption (2.3.4) on J and (2.3.7), we see that $(n-1)[Q_n(u,v) - Q_n^*(u,v)]$ is $O(1)$ and since $E(Y^2) < \infty$ LDCT applies in (2.3.13). Hence an

application of LDCT along with lemma 2.3.1 yields

$$\begin{aligned}
 (2.3.14) \quad \lim_{n \rightarrow \infty} \text{Var}(B_k) &= \iiint J_1'(G(u_1), F(v)) J_1'(G(u_2), F(v_2)) [G(u_1 \wedge u_2) - G(u_1)G(u_2)] \\
 &\quad v_1 v_2 dH(u_1, v_1) dH(u_2, v_2).
 \end{aligned}$$

Similarly, an application of LDCT along with lemma 2.3.1, also gives

$$\begin{aligned}
 (2.3.15) \quad \lim_{n \rightarrow \infty} \text{Var}(C_k) &= \lim_{n \rightarrow \infty} \iiint (n-1) [\bar{Q}_n(u_1, v_1) - Q_n(u_1, v_1) + Q_n^*(u_1, v_1) - \bar{Q}_n^*(u_1, v_1)] \\
 &\quad \times (n-1) [\bar{Q}_n(u_2, v_2) - Q_n(u_2, v_2) + Q_n^*(u_2, v_2) - \bar{Q}_n^*(u_2, v_2)] \\
 &\quad \times [H(u_1 \wedge u_2, v_1 \wedge v_2) - H(u_1, v_1)H(u_2, v_2)] \\
 &\quad \times v_1 v_2 dH(u_1, v_1) dH(u_2, v_2) \\
 &= 0,
 \end{aligned}$$

$$\begin{aligned}
 (2.3.16) \quad \lim_{n \rightarrow \infty} \text{Var}(D_k) &= \lim_{n \rightarrow \infty} \iiint (n-1) [\bar{Q}_n^*(u_1, v_1) - Q_n^*(u_1, v_1)] \\
 &\quad \times (n-1) [\bar{Q}_n^*(u_2, v_2) - Q_n^*(u_2, v_2)] \\
 &\quad \times [F(v_1 \wedge v_2) - F(v_1)F(v_2)] v_1 v_2 dH(u_1, v_1) dH(u_2, v_2) \\
 &= \iiint J_2'(G(u_1), F(v_1)) J_2'(G(u_2), F(v_2)) \\
 &\quad \times [F(v_1 \wedge v_2) - F(v_1)F(v_2)] v_1 v_2 dH(u_1, v_1) dH(u_2, v_2).
 \end{aligned}$$

Now from (2.3.11), we have

$$\begin{aligned}
E(A_k \cdot B_k) &= \iint [v_1 \tilde{Q}_n(u_1, v_1)] [(n-1) \iint \{Q_n(u_2, v_2) - Q_n^*(u_2, v_2)\} I(u_1 \leq u_2)] \\
&\quad \times dH(u_1, v_1) v_2 dH(u_2, v_2) \\
&= \iiint \tilde{Q}_n(u_1, v_1) (n-1) [Q_n(u_2, v_2) - Q_n^*(u_2, v_2)] I(u_1 \leq u_2) v_1 v_2 \\
&\quad dH(u_1, v_1) dH(u_2, v_2),
\end{aligned}$$

and

$$\begin{aligned}
E(A_k) \cdot E(B_k) &= \left(\iint v_1 \tilde{Q}_n(u_1, v_1) dH(u_1, v_1) \right) \\
&\quad \times \left(\iint (n-1) [Q_n(u_2, v_2) - Q_n^*(u_2, v_2)] G(u_2) v_2 dH(u_2, v_2) \right) \\
&= \iiint \tilde{Q}_n(u_1, v_1) (n-1) [Q_n(u_2, v_2) - Q_n^*(u_2, v_2)] G(u_2) v_1 v_2 \\
&\quad \times dH(u_1, v_1) dH(u_2, v_2).
\end{aligned}$$

Again applying LDCT and Lemma 2.3.1 yields

$$\begin{aligned}
(2.3.17) \quad \lim_{H \rightarrow \infty} \text{Cov}(A_k, B_k) &= \lim_{H \rightarrow \infty} \iiint \tilde{Q}_n(u_1, v_1) (n-1) [Q_n(u_2, v_2) - Q_n^*(u_2, v_2)] \\
&\quad \times [I(u_1 \leq u_2) - G(u_2)] v_1 v_2 dH(u_1, v_1) dH(u_2, v_2) \\
&= \iiint J(G(u_1), F(v_1)) J_1'(G(u_2), F(v_2)) \\
&\quad \times [I(u_1 \leq u_2) - G(u_2)] v_1 v_2 dH(u_1, v_1) dH(u_2, v_2).
\end{aligned}$$

On the same lines, we can see that

$$\begin{aligned}
(2.3.18) \quad \lim_{H \rightarrow \infty} \text{Cov}(A_k, D_k) &= \iiint J(G(u_1), F(v_1)) J_2'(G(u_2), F(v_2)) \\
&\quad \times [I(v_1 \leq v_2) - F(v_2)] v_1 v_2 dH(u_1, v_1) dH(u_2, v_2),
\end{aligned}$$

(2.3.19)

$$\lim_{H \rightarrow \infty} \text{Cov}(A_k, C_k) = 0$$

$$\lim_{H \rightarrow \infty} \text{Cov}(B_k, C_k) = 0$$

$$\lim_{H \rightarrow \infty} \text{Cov}(C_k, D_k) = 0$$

$$\begin{aligned}
 (2.3.20) \quad \lim_{H \rightarrow \infty} \text{Cov}(B_k, D_k) &= \lim_{H \rightarrow \infty} \iiint (n-1) [Q_n(u_1, v_1) - Q_n^*(u_1, v_1)] \\
 &\quad \times (n-1) [\bar{Q}_n^*(u_2, v_2) - Q_n^*(u_2, v_2)] [H(u_1, v_2) - G(u_1)F(v_2)] \\
 &\quad \times v_1 v_2 dH(u_1, v_1) dH(u_2, v_2) \\
 &= \iiint J_1'(G(u_1), F(v_1)) J_2'(Q(u_2), F(v_2)) \\
 &\quad \times [H(u_1, v_2) - G(u_1)F(v_2)] v_1 v_2 dH(u_1, v_1) dH(u_2, v_2).
 \end{aligned}$$

The lemma now follows by using (2.3.12) and (2.3.14) to (2.3.20) with

(2.3.10). $\square$

CHAPTER III

ASYMPTOTIC NORMALITY OF S_n

In this chapter we prove the asymptotic normality of S_n as defined by (2.1.1), in a properly normalized form. This we shall do by first establishing the asymptotic equivalence in mean square of T_n and $\hat{T}_n$ and then proving the asymptotic normality of $\hat{T}_n$ where T_n and $\hat{T}_n$ are as defined in section 2.2 (also given by (3.1.1) and (3.1.2) below). A sequence of rv's $\{X_n\}$ is asymptotically normal with mean μ_n and variance σ_n^2 if $\sigma_n > 0$ for all n sufficiently large and

$$\frac{X_n - \mu_n}{\sigma_n} \rightarrow N(0,1) \text{ in distribution as } n \rightarrow \infty$$

$$\text{i.e. } \left| P\left(\frac{X_n - \mu_n}{\sigma_n} \leq t\right) - \Phi(t) \right| \rightarrow 0, \text{ as } n \rightarrow \infty \text{ for } -\infty < t < \infty,$$

where $\Phi(t)$ is the standard normal distribution function.

In section 3.1. we state our main results and establish the asymptotic equivalence of T_n and $\hat{T}_n$ while using the results obtained in Chapter II. The proofs of the main results are given in sections 3.2 and 3.3.

3.1 ASYMPTOTIC EQUIVALENCE OF S_n AND $\hat{S}_n$. Let S_n and $\hat{S}_n$ be as given by (2.1.1) and (2.3.2) respectively and let

$$(3.1.1) \quad T_n = n^{1/2} (S_n - E(S_n))$$

$$(3.1.2) \quad \hat{T}_n = n^{1/2} (\hat{S}_n - E(\hat{S}_n))$$

as defined in section 2.2. In this section we establish the asymptotic equivalence of T_n and $\hat{T}_n$, in mean square. But we first state the main result of this chapter, namely, the asymptotic normality of T_n as Theorem 3.1.1 and the two important special cases considered by Yang [22] and Stigler [17] as Theorem 3.1.2 below.

Theorem 3.1.1. Assume that $E(X^2) < \infty$ and $E(Y^2) < \infty$. Let $J(u,v)$, $0 \leq u, v \leq 1$, be a bivariate score function satisfying assumptions (2.3.3) and (2.3.4). Then, if H is continuous and $\sigma^2(J,H) > 0$

$$\left(\frac{S_n - E(S_n)}{\sigma(S_n)} \right) \rightarrow N(0,1),$$

as $n \rightarrow \infty$, where $\sigma^2(J,H) = \lim_{n \rightarrow \infty} n \text{Var}(S_n)$ is as given by (2.3.9).

Theorem 3.1.2. If the function J in Theorem 3.1.1 above satisfies

- (a) $J(u,v) = J(v)$, $0 \leq v \leq 1$, then the assumptions and conclusions of Theorem 3.1.1 reduce to those of Theorem 2 of Stigler [17], with $\sigma^2(J,F)$ given by

$$(3.1.3) \quad \sigma^2(J,F) = \iint J(F(v_1))J(F(v_2))[F(v_1 \wedge v_2) - F(v_1)F(v_2)]dv_1dv_2;$$

- (b) $J(u,v) = J(u)$, $0 \leq u \leq 1$, then the assumptions and conclusions of Theorem 3.1.1 reduce to those of Theorem 1 of Yang [22], under the additional assumption that $m(x) = E[Y|X = x]$, $-\infty < x < \infty$ is a function of bounded variation, with $\sigma^2(J,H) \equiv \sigma^2(J,G)$ given by

$$\begin{aligned}
 (3.1.4) \quad \sigma^2(J, H) &= \int J^2(G(u)) \sigma^2(u) dG(u) \\
 &+ \iint [G(u_1) \Lambda u_2 - G(u_1) G(u_2)] \\
 &\quad \times J(G(u_1)) J(G(u_2)) dm(u_1) dm(u_2)
 \end{aligned}$$

where $\sigma^2(u) = \text{Var}(Y|X = u)$.

COROLLARY 3.1.1. Under the conditions of Theorem 3.1.1

$$\frac{S_n - \mu(J, H)}{\sigma(S_n)} \rightarrow N(0, 1) \text{ in distribution, as } n \rightarrow \infty$$

where

$$\mu(J, H) = \iint J(G(u), F(v)) v dH(u, v).$$

The proofs of Theorem 3.1.1 and Corollary 3.1.1 are given in section 3.2 and the proof of Theorem 3.1.2 is given in section 3.3.

LEMMA 3.1.1. Suppose assumptions (2.2.3) and (2.2.4) on J are satisfied. We then have

$$\lim_{n \rightarrow \infty} \text{Var}(T_n) = \lim_{n \rightarrow \infty} n \text{Var}(S_n) = \sigma^2(J, H)$$

where $\sigma^2(J, H)$ is as given by (2.3.9).

PROOF. First note that

$$\begin{aligned}
 (3.1.5) \quad n \text{Var}(S_n) &= n^{-1} \left[\sum_{i=1}^n \text{Var}\left(J\left(\frac{1}{n}, \frac{R_{in}}{n}\right) Y_{[i:n]}\right) \right. \\
 &+ \sum_{i < i'} \text{Cov}\left(J\left(\frac{1}{n}, \frac{R_{in}}{n}\right) Y_{[i:n]}, J\left(\frac{1}{n}, \frac{R_{i'n}}{n}\right) Y_{[i':n]}\right) \\
 &\left. + \sum_{i > i'} \text{Cov}\left(J\left(\frac{1}{n}, \frac{R_{in}}{n}\right) Y_{[i:n]}, J\left(\frac{1}{n}, \frac{R_{i'n}}{n}\right) Y_{[i':n]}\right) \right].
 \end{aligned}$$

Now setting $r(Y_j) = \text{rank of } Y_j \text{ among the } n \text{ } Y\text{'s}$, $r(X_j) = \text{rank of } X_j \text{ among the } n \text{ } X\text{'s}$, we can see that

$$(3.1.6) \quad E\left[J\left(\frac{1}{n}, \frac{R_{in}}{n}\right)Y_{[i:n]}\right] = nE\left(J\left(\frac{1}{n}, \frac{r(Y_n)}{n}\right)Y_n I(r(X_n) = i)\right) \\ = n \iint \left[\sum_{s=1}^n J\left(\frac{1}{n}, \frac{s}{n}\right) p_{i-1,s-1}^{(n-1)}(u,v) \right] v dH(u,v),$$

where $p_{i,s}^{(n)}(u,v)$ is defined by (2.2.3), so that

$$(3.1.7) \quad \text{Var}\left(J\left(\frac{1}{n}, \frac{R_{in}}{n}\right)Y_{[i:n]}\right) = n \iint \left[\sum_{s=1}^n J^2\left(\frac{1}{n}, \frac{s}{n}\right) p_{i-1,s-1}^{(n-1)}(u,v) \right] v^2 dH(u,v) \\ - n^2 \left(\iint \left[\sum_{s=1}^n J\left(\frac{1}{n}, \frac{s}{n}\right) p_{i-1,s-1}^{(n-1)}(u,v) \right] v dH(u,v) \right)^2 \\ = n \iint \sum_{s=1}^n J^2\left(\frac{1}{n}, \frac{s}{n}\right) p_{i-1,s-1}^{(n-1)}(u,v) v^2 dH(u,v) \\ - n^2 \iiint \left[\sum_{s=1}^n J\left(\frac{1}{n}, \frac{s}{n}\right) p_{i-1,s-1}^{(n-1)}(u_1, v_1) \right] \\ \times \left[\sum_{s'=1}^n J\left(\frac{1}{n}, \frac{s'}{n}\right) p_{i-1,s'-1}^{(n-1)}(u_2, v_2) \right] v_1 v_2 dH(u_1, v_1) \\ dH(u_2, v_2).$$

Further for $i < i'$

$$\begin{aligned}
(3.1.8) \quad & E\left(J\left(\frac{i}{n}, \frac{R_{in}}{n}\right) Y_{[i:n]} J\left(\frac{i'}{n}, \frac{R_{i'n}}{n}\right) Y_{[i':n]}\right) \\
&= n(n-1) E\left[J\left(\frac{i}{n}, \frac{r(Y_n)}{n}\right) Y_n J\left(\frac{i'}{n}, \frac{r(Y_{n-1})}{n}\right) Y_{n-1}\right. \\
&\quad \left. I(r(X_n) = i, r(X_{n-1}) = i')\right] \\
&= n(n-1) E\left[J\left(\frac{i}{n}, \frac{r(Y_n)}{n}\right) Y_n J\left(\frac{i'}{n}, \frac{r(Y_{n-1})}{n}\right) Y_{n-1} I(Y_n < Y_{n-1})\right. \\
&\quad \left. I(r(X_n) = i, r(X_{n-1}) = i')\right] \\
&\quad + n(n-1) E\left[J\left(\frac{i}{n}, \frac{r(Y_n)}{n}\right) Y_n J\left(\frac{i'}{n}, \frac{r(Y_{n-1})}{n}\right) Y_{n-1} I(Y_n > Y_{n-1})\right. \\
&\quad \left. I(r(X_n) = i, r(X_{n-1}) = i')\right] \\
&= n(n-1) \iiint\limits_{\substack{u_1 \leq u_2 \\ v_1 \leq v_2}} \left[\sum_{s < s'} J\left(\frac{i}{n}, \frac{s}{n}\right) J\left(\frac{i'}{n}, \frac{s'}{n}\right) p_{is i's'}^{11}(u_1, v_1, u_2, v_2) \right] v_1 v_2 \\
&\quad \cdot dH(u_1, v_1) dH(u_2, v_2) \\
&\quad + n(n-1) \iiint\limits_{\substack{u_1 < u_2 \\ v_1 > v_2}} \left[\sum_{s > s'} J\left(\frac{i}{n}, \frac{s}{n}\right) J\left(\frac{i'}{n}, \frac{s'}{n}\right) p_{is i's'}^{12}(u_1, v_1, u_2, v_2) \right] v_1 v_2 \\
&\quad \cdot dH(u_1, v_1) dH(u_2, v_2)
\end{aligned}$$

where

$$\begin{aligned}
(3.1.9) \quad & p_{is i's'}^{11}(u_1, v_1, u_2, v_2) = P[\text{exactly } (i-1) \text{ X's } < u_1, (i'-2) \text{ X's } < u_2, \\
&\quad (s-1) \text{ Y's } < v_1, (s'-2) \text{ Y's } < v_2, \\
&\quad \text{out of } (n-2) \text{ iid } (X, Y) \text{ pairs}], \\
&\quad \text{for } u_1 < u_2, v_1 < v_2; i < i', s < s'
\end{aligned}$$

and

$$\begin{aligned}
 (3.1.10) \quad p_{isi's'}^{12}(u_1, v_1, u_2, v_2) &= P[\text{exactly } (i-1) \text{ X's } < u_1, (i'-2) \text{ X's } < u_2, \\
 &\quad (s'-1) \text{ Y's } < v_2, (s-2) \text{ Y's } < v_1 \text{ out of } (n-2) \\
 &\quad \text{iid (X,Y) pairs}] \\
 &\quad \text{for } u_1 < u_2, v_1 > v_2; i < i', s > s'.
 \end{aligned}$$

Observe that (3.1.10) is easily obtained from (3.1.9) by interchanging v_1 and v_2 and s and s' . (The explicit expressions of these probabilities can be obtained using the multinomial distribution in the same way as the probabilities in (2.2.3) are obtained using the Binomial distribution. However, since we do not require the explicit expressions in our arguments they are omitted). Now it follows from (3.1.6) and (3.1.8) that for $i < i'$

$$\begin{aligned}
 (3.1.11) \quad \text{Cov}\left(J\left(\frac{i}{n}, \frac{R_{in}}{n}\right)Y_{[i:n]}, J\left(\frac{i'}{n}, \frac{R_{i'n}}{n}\right)Y_{[i':n]}\right) \\
 = n(n-1) \left[\iiint_{\substack{u_1 < u_2 \\ v_1 < v_2}} \left[\sum_{s < s'} J\left(\frac{i}{n}, \frac{s}{n}\right) J\left(\frac{i'}{n}, \frac{s'}{n}\right) p_{isi's'}^{11}(u_1, v_1, u_2, v_2) \right] \right. \\
 \quad \times v_1 v_2 dH(u_1, v_1) dH(u_2, v_2) \\
 + \iiint_{\substack{u_1 < u_2 \\ v_1 > v_2}} \left[\sum_{s > s'} J\left(\frac{i}{n}, \frac{s}{n}\right) J\left(\frac{i'}{n}, \frac{s'}{n}\right) p_{isi's'}^{12}(u_1, v_1, u_2, v_2) \right] \\
 \quad \times v_1 v_2 dH(u_1, v_1) dH(u_2, v_2) \Big] \\
 - n^2 \iiint \left[\sum_{s=1}^n J\left(\frac{i}{n}, \frac{s}{n}\right) p_{i-1, s-1}^{(n-1)}(u_1, v_1) \right] \\
 \times \left[\sum_{s'=1}^n J\left(\frac{i'}{n}, \frac{s'}{n}\right) p_{i'-1, s'-1}^{(n-1)}(u_2, v_2) \right] v_1 v_2 dH(u_1, v_1) dH(u_2, v_2),
 \end{aligned}$$

and similarly for $i > i'$ we obtain

$$\begin{aligned}
 (3.1.12) \quad \text{Cov}\left(J\left(\frac{i}{n}, \frac{R_{in}}{n}\right)Y_{[i:n]}, J\left(\frac{i'}{n}, \frac{R_{i'n}}{n}\right)Y_{[i':n]}\right) \\
 = n(n-1) \iiint \left[\sum_{\substack{u_1 > u_2 \\ v_1 < v_2}} \sum_{s < s'} J\left(\frac{i}{n}, \frac{s}{n}\right) J\left(\frac{i's'}{n}, \frac{s'}{n}\right) p_{isi's'}^{21}(u_1, v_1, u_2, v_2) \right] \\
 \quad \times v_1 v_2 dH(u_1, v_1) dH(u_2, v_2) \\
 + n(n-1) \iiint \left[\sum_{\substack{u_1 > u_2 \\ v_1 > v_2}} \sum_{s > s'} J\left(\frac{i}{n}, \frac{s}{n}\right) J\left(\frac{i's'}{n}, \frac{s'}{n}\right) p_{isi's'}^{22}(u_1, v_1, u_2, v_2) \right] \\
 \quad \times v_1 v_2 dH(u_1, v_1) dH(u_2, v_2) \\
 - n^2 \iiint \left[\sum_{s=1}^n J\left(\frac{i}{n}, \frac{s}{n}\right) p_{i-1, s-1}^{(n-1)}(u_1, v_1) \right] \\
 \quad \times \left[\sum_{s'=1}^n J\left(\frac{i'}{n}, \frac{s'}{n}\right) p_{i'-1, s'-1}^{(n-1)}(u_2, v_2) \right] v_1 v_2 dH(u_1, v_1) dH(u_2, v_2),
 \end{aligned}$$

where the probabilities $p_{isi's'}^{21}(u_1, v_1, u_2, v_2)$ are defined as those obtained by interchanging u_1 and u_2 and i and i' in the definitions of $p_{isi's'}^{12}(u_1, v_1, u_2, v_2)$ and $p_{isi's'}^{22}(u_1, v_1, u_2, v_2)$ (given by (3.1.9) and (3.1.10)) respectively. Now using (3.1.7), (3.1.11) and (3.1.12) in (3.1.5) we obtain

$$\begin{aligned}
 (3.1.13) \quad n \text{Var}(S_n) = \iint \left[\sum_{i=1}^n \sum_{s=1}^n J^2\left(\frac{i}{n}, \frac{s}{n}\right) p_{i-1, s-1}^{(n-1)}(u, v) \right] v^2 dH(u, v) \\
 + (n-1) \iiint \left[\sum_{\substack{u_1 < u_2 \\ v_1 < v_2}} \sum_{i < i'} \sum_{s < s'} J\left(\frac{i}{n}, \frac{s}{n}\right) J\left(\frac{i's'}{n}, \frac{s'}{n}\right) p_{isi's'}^{11}(u_1, v_1, u_2, v_2) \right] \\
 \quad \times v_1 v_2 dH(u_1, v_1) dH(u_2, v_2)
 \end{aligned}$$

continued

$$+ (n-1) \iiint\limits_{\substack{u_1 < u_2 \\ v_1 > v_2}} \left[\sum_{1 < 1'} \sum_{s > s'} J\left(\frac{1}{n}, \frac{s}{n}\right) J\left(\frac{1'}{n}, \frac{s'}{n}\right) p_{1s1's'}^{12}(u_1, v_1, u_2, v_2) \right] \\ \times v_1 v_2 dH(u_1, v_1) dH(u_2, v_2)$$

$$+ (n-1) \iiint\limits_{\substack{u_1 > u_2 \\ v_1 < v_2}} \left[\sum_{1 > 1'} \sum_{s < s'} J\left(\frac{1}{n}, \frac{s}{n}\right) J\left(\frac{1'}{n}, \frac{s'}{n}\right) p_{1s1's'}^{21}(u_1, v_1, u_2, v_2) \right] \\ \times v_1 v_2 dH(u_1, v_1) dH(u_2, v_2)$$

$$+ (n-1) \iiint\limits_{\substack{u_1 > u_2 \\ v_1 > v_2}} \left[\sum_{1 > 1'} \sum_{s > s'} J\left(\frac{1}{n}, \frac{s}{n}\right) J\left(\frac{1'}{n}, \frac{s'}{n}\right) p_{1s1's'}^{22}(u_1, v_1, u_2, v_2) \right] \\ \times v_1 v_2 dH(u_1, v_1) dH(u_2, v_2)$$

$$- n \iiint\limits_{\substack{1=1 \\ 1'=1 \\ s=1 \\ s'=1}}^n \left[\sum_{1=1}^n \sum_{1'=1}^n \sum_{s=1}^n \sum_{s'=1}^n J\left(\frac{1}{n}, \frac{s}{n}\right) J\left(\frac{1'}{n}, \frac{s'}{n}\right) p_{1-1, s-1}^{(n-1)}(u_1, v_1) p_{1'-1, s'-1}^{(n-1)}(u_2, v_2) \right] \\ \times v_1 v_2 dH(u_1, v_1) dH(u_2, v_2).$$

By Taylor's expansion of $J\left(\frac{1}{n}, \frac{s}{n}\right)$ around the point $(G(u_1), F(v_1))$ and that of $J\left(\frac{1'}{n}, \frac{s'}{n}\right)$ around $(G(u_2), F(v_2))$, we may write

$$(3.1.14) \quad J\left(\frac{1}{n}, \frac{s}{n}\right) = J(G(u_1), F(v_1)) + \left(\frac{1}{n} - G(u_1)\right) J_1'(G(u_1), F(v_1)) \\ + \left(\frac{s}{n} - F(v_1)\right) J_2'(G(u_1), F(v_1)) \\ + \frac{\left(\frac{1}{n} - G(u_1)\right)^2}{2} J_1''(G(u_1), F(v_1)) \\ + \frac{\left(\frac{s}{n} - F(v_1)\right)^2}{2} J_2''(G(u_1), F(v_1)) \\ + \left(\frac{1}{n} - G(u_1)\right) \left(\frac{s}{n} - F(v_1)\right) J_{12}''(G(u_1), F(v_1)) + R_n$$

and

$$\begin{aligned}
 (3.1.15) \quad J\left(\frac{i'}{n}, \frac{s'}{n}\right) &= J(G(u_2), F(v_2)) + \left(\frac{i'}{n} - G(u_2)\right) J_1'(G(u_2), F(v_2)) \\
 &\quad + \left(\frac{s'}{n} - F(v_2)\right) J_2'(G(u_2), F(v_2)) \\
 &\quad + \frac{\left(\frac{i'}{n} - G(u_2)\right)^2}{2} J_1''(G(u_2), F(v_2)) \\
 &\quad + \frac{\left(\frac{s'}{n} - F(v_2)\right)^2}{2} J_2''(G(u_2), F(v_2)) \\
 &\quad + \left(\frac{i'}{n} - G(u_2)\right) \left(\frac{s'}{n} - F(v_2)\right) J_{12}''(G(u_2), F(v_2)) + R_n',
 \end{aligned}$$

where R_n and R_n' include all higher order terms.

Also observe that

$$(3.1.16) \quad \sum_{i=1}^n \sum_{s=1}^n p_{i-1, s-1}^{(n-1)}(u, v) = 1$$

$$\begin{aligned}
 \sum_{i < i'} \sum_{s < s'} \sum_{i_1 s_1} p_{i_1 s_1}^{11}(u_1, v_1, u_2, v_2) &= \sum_{i < i'} \sum_{s > s'} \sum_{i_1 s_1} p_{i_1 s_1}^{12}(u_1, v_1, u_2, v_2) \\
 &= \sum_{i > i'} \sum_{s < s'} \sum_{i_1 s_1} p_{i_1 s_1}^{21}(u_1, v_1, u_2, v_2) \\
 &= \sum_{i > i'} \sum_{s > s'} \sum_{i_1 s_1} p_{i_1 s_1}^{22}(u_1, v_1, u_2, v_2) \\
 &= 1.
 \end{aligned}$$

and for fixed $i = 1, 2, \dots, n-1$,

$$\sum_{i'=1+1}^n \sum_{s < s'} p_{i s i' s'}^{11}(u_1, v_1, u_2, v_2) = P[\text{exactly } (i-1) \text{ X's } < u_1 \text{ out of } (n-2) \text{ i.i.d. pairs}] \\ = p_{i-1}^{(n-2)}(u_1),$$

for fixed i, i' with $i < i', i = 1, 2, \dots, n-1, i' = 2, 3, \dots, n$,

$$\sum_{s < s'} p_{i s i' s'}^{11}(u_1, v_1, u_2, v_2) = P[\text{exactly } (i-1) \text{ X's } < u_1, \\ (i'-2) \text{ X's } < u_2 \text{ out of } (n-2) \text{ i.i.d. pairs}] \\ = p_{i i'}^{11}(u_1, u_2),$$

for fixed $i', s', i' = 2, 3, \dots, n, s' = 2, 3, \dots, n$

$$\sum_{i < i' s < s'} p_{i s i' s'}^{11}(u_1, v_1, u_2, v_2) = P[\text{exactly } (i'-2) \text{ X's } < u_2, \\ (s'-2) \text{ Y's } < v_2 \text{ out of } (n-2) \text{ i.i.d. pairs}] \\ = p_{i' s'}^{11}(u_2, v_2),$$

and for fixed $i = 1, 2, \dots, n$,

$$\sum_{s=1}^n p_{i-1, s-1}^{(n-1)}(u, v) = P[\text{exactly } (i-1) \text{ X's } < u \text{ out of } (n-1) \text{ i.i.d. pairs}], \\ = p_{i-1}^{(n-1)}(u)$$

and similar relations hold for the probabilities $p_{i s i' s'}^{12}(u_1, v_1, u_2, v_2)$,

$p_{i s i' s'}^{21}(u_1, v_1, u_2, v_2)$ and $p_{i s i' s'}^{22}(u_1, v_1, u_2, v_2)$. Now using the expansions given

by (3.1.14) and (3.1.15) in place of $J(\frac{i}{n}, \frac{s}{n})$ and $J(\frac{i'}{n}, \frac{s'}{n})$, respectively,

in (3.1.13) and also the relations of the type given by (3.1.16) therein, we

obtain the resulting expression (on collecting similar terms) as the sum of the

expressions $\textcircled{F}, \textcircled{G}, \textcircled{H}, \textcircled{I}, \textcircled{J}, \textcircled{K}, \textcircled{L}, \textcircled{M}$

given below:

The sum of terms involving $J^2(G(u), F(v))$, viz.,

$$(3.1.17) \quad \textcircled{F} = \iint J^2(G(u), F(v)) v^2 dH(u, v).$$

The sum of terms involving $J(G(u_1), F(v_1))J(G(u_2), F(v_2))$, viz.,

$$(3.1.18) \quad \begin{aligned} \textcircled{G} &= (n-1) \iiint J(G(u_1), F(v_1))J(G(u_2), F(v_2)) v_1 v_2 dH(u_1, v_1) dH(u_2, v_2) \\ &\quad - n \iiint J(G(u_1), F(v_1))J(G(u_2), F(v_2)) v_1 v_2 dH(u_1, v_1) dH(u_2, v_2) \\ &= - \iiint J(G(u_1), F(v_1))J(G(u_2), F(v_2)) v_1 v_2 dH(u_1, v_1) dH(u_2, v_2). \end{aligned}$$

Let $T_1 = \sum_{i=1}^{n-2} I(X_i \leq u_1) = \{\text{\# of } X\text{'s } \leq u_1 \text{ out of } (n-2) \text{ iid observations}\}.$

Hence T_1 is Binomial $(n-2, G(u_1))$ and

$$\sum_{i=0}^{n-2} [1 - (n-2)G(u_1)] p_1^{(n-2)}(u_1) = E(T_1 - ET_1) = 0.$$

This together with (3.1.16) yields the relations

$$(3.1.19) \quad \sum_{i < 1'} \sum_{s < s'} \left(\frac{1}{n} - G(u_1) \right) p_{1s1's'}^{11}(u_1, v_1, u_2, v_2) = \sum_{i=1}^{n-1} \left(\frac{1}{n} - G(u_1) \right) p_{i-1}^{(n-2)}(u_1) \\ = \frac{1}{n} (1 - 2G(u_1)),$$

$$\sum_{i < 1'} \sum_{s > s'} \left(\frac{1}{n} - G(u_1) \right) p_{1s1's'}^{12}(u_1, v_1, u_2, v_2) = \sum_{i=1}^{n-1} \left(\frac{1}{n} - G(u_1) \right) p_{i-1}^{(n-2)}(u_1) \\ = \frac{1}{n} (1 - 2G(u_1)),$$

$$\sum_{i>1} \sum_{s<s'} \sum_{s'} \left(\frac{1}{n} - G(u_1) \right) p_{1s1's'}^{12}(u_1, v_1, u_2, v_2) = \sum_{i=2}^n \left(\frac{1}{n} - G(u_1) \right) p_{1-2}^{(n-2)}(u_1) \\ = \frac{1}{n} (2-2G(u_1)),$$

$$\sum_{i>1} \sum_{s>s'} \sum_{s'} \left(\frac{1}{n} - G(u_1) \right) p_{1s1's'}^{22}(u_1, v_1, u_2, v_2) = \sum_{i=2}^n \left(\frac{1}{n} - G(u_1) \right) p_{1-2}^{(n-2)}(u_1) \\ = \frac{1}{n} (2-2G(u_1)),$$

and

$$\sum_{i=1} \sum_{i'} \sum_{s} \sum_{s'} \left(\frac{1}{n} - G(u_1) \right) p_{i-1, s-1}^{(n-1)}(u_1, v_1) p_{i'-1, s'-1}^{(n-1)}(u_2, v_2) \\ = \sum_{i=1} \left(\frac{1}{n} - G(u_1) \right) p_{i-1}^{(n-1)}(u_1) = \frac{1}{n} (1-G(u_1)).$$

Using (3.1.19) we obtain the sum of terms involving

$J_1'(G(u_1), F(v_1)) J(G(u_2), F(v_2))$ as

$$(3.1.20) \quad \textcircled{H} = (n-1) \int_{u_1 < u_2} \int_{v_1 < v_2} J_1'(G(u_1), F(v_1)) J(G(u_2), F(v_2)) \\ \times \frac{1}{n} (1-2G(u_1)) v_1 v_2 dH(u_1, v_1) dH(u_2, v_2) \\ + (n-1) \int_{u_1 < u_2} \int_{v_1 > v_2} J_1'(G(u_1), F(v_1)) J(G(u_2), F(v_2)) \\ \times \frac{1}{n} (1-2G(u_1)) v_1 v_2 dH(u_1, v_1) dH(u_2, v_2) \\ + (n-1) \int_{u_1 > u_2} \int_{v_1 < v_2} J_1'(G(u_1), F(v_1)) J(G(u_2), F(v_2)) \\ \times \frac{1}{n} (2-2G(u_1)) v_1 v_2 dH(u_1, v_1) dH(u_2, v_2)$$

continued

$$\begin{aligned}
& + (n-1) \iiint_{\substack{u_1 > u_2 \\ v_1 > v_2}} J_1'(G(u_1), F(v_1)) J(G(u_2), F(v_2)) \\
& \quad \times \frac{1}{n} (2-2G(u_1)) v_1 v_2 dH(u_1, v_1) dH(u_2, v_2) \\
& - n \iiint J_1'(G(u_1), F(v_1)) J(G(u_2), F(v_2)) \\
& \quad \times \frac{1}{n} (1-G(u_1)) v_1 v_2 dH(u_1, v_1) dH(u_2, v_2) \\
& = \frac{n-1}{n} \iiint_{u_1 < u_2} J_1'(G(u_1), F(v_1)) J(G(u_2), F(v_2)) \\
& \quad \times (-G(u_1)) v_1 v_2 dH(u_1, v_1) dH(u_2, v_2) \\
& + \frac{n-1}{n} \iiint_{u_1 > u_2} J_1'(G(u_1), F(v_1)) J(G(u_2), F(v_2)) \\
& \quad \times (1-G(u_1)) v_1 v_2 dH(u_1, v_1) dH(u_2, v_2) \\
& - \frac{1}{n} \iiint J_1'(G(u_1), F(v_1)) (G(u_2), F(v_2)) (1-G(u_1)) \\
& \quad \times v_1 v_2 dH(u_1, v_1) dH(u_2, v_2) \\
& = \frac{n-1}{n} \iiint J_1'(G(u_1), F(v_1)) J(G(u_2), F(v_2)) [I(u_1 > u_2) - G(u_1)] \\
& \quad \times v_1 v_2 dH(u_1, v_1) dH(u_2, v_2) \\
& + O(1/n).
\end{aligned}$$

A similar argument reduces the sum of the terms involving

$J_1'(G(u_2), F(v_2)) J(G(u_1), F(v_1))$ to

$$\begin{aligned}
(3.1.21) \quad \textcircled{I} & = \frac{n-1}{n} \iiint J_1'(G(u_2), F(v_2)) J(G(u_1), F(v_1)) [I(u_1 < u_2) - G(u_2)] \\
& \quad \times v_1 v_2 dH(u_1, v_1) dH(u_2, v_2) \\
& + O(1/n).
\end{aligned}$$

(Note that the expression in (3.1.21) is exactly the same as that of (3.1.20) with (u_1, v_1) and (u_2, v_2) interchanged) and the sum of the terms involving $J_2'(G(u_1), F(v_1))J(G(u_2), F(v_2))$ and $J_2'(G(u_2), F(v_2))J(G(u_1), F(v_1))$ to

$$(3.1.22) \quad \textcircled{J} = \frac{2(n-1)}{n} \iiint J_2'(G(u_2), F(v_2))J(G(u_1), F(v_1)) \\ \times [I(v_1 < v_2) - F(v_2)] v_1 v_2 dH(u_1, v_1) dH(u_2, v_2) \\ + O(1/n).$$

Since

$$(3.1.23) \quad \sum_{i < i'} \sum_{s < s'} \left(\frac{i}{n} - G(u_1) \right) \left(\frac{i'}{n} - G(u_2) \right) p_{is i's'}^{11}(u_1, v_1, u_2, v_2) \\ = \sum_{i < i'} \sum_{s > s'} \left(\frac{i}{n} - G(u_1) \right) \left(\frac{i'}{n} - G(u_2) \right) p_{is i's'}^{12}(u_1, v_1, u_2, v_2) \\ = \sum_{i < i'} \left(\frac{i}{n} - G(u_1) \right) \left(\frac{i'}{n} - G(u_2) \right) p_{ii'}^{11}(u_1, u_2) \\ = \sum_{i=1}^{n-1} \sum_{i'=i+1}^n \left(\frac{i}{n} - G(u_1) \right) \left(\frac{i'}{n} - G(u_2) \right) p_{ii'}^{11}(u_1, u_2) \\ = \frac{1}{n^2} \sum_{i=1}^{n-1} \sum_{i'=i+1}^{n-1} [(i-1) - (n-2)G(u_1)][(i'-2) - (n-2)G(u_2)] p_{ii'}^{11}(u_1, u_2) \\ + \frac{(1-2G(u_1))(2-2G(u_2))}{n^2} \\ = \frac{1}{n^2} \text{Cov} \left(\sum_{i=1}^{n-2} I(x_{i+1} \leq u_1), \sum_{j=1}^{n-2} I(x_{j+1} \leq u_2) \right) \\ + O\left(\frac{1}{n^2}\right)$$

continued

$$= \frac{n-2}{n^2} [G(u_1 \wedge u_2) - G(u_1)G(u_2)] + o\left(\frac{1}{n^2}\right),$$

and similarly

$$\begin{aligned} (3.1.24) \quad & \sum_{i>1'} \sum_{s<s'} \sum \left(\frac{1}{n} - G(u_1)\right) \left(\frac{1'}{n} - G(u_2)\right) p_{1s1's'}^{21}(u_1, v_1, u_2, v_2) \\ &= \sum_{i>1'} \sum_{s>s'} \sum \left(\frac{1}{n} - G(u_1)\right) \left(\frac{1'}{n} - G(u_2)\right) p_{1s1's'}^{22}(u_1, v_1, u_2, v_2) \\ &= \frac{n-2}{n^2} [G(u_1 \wedge u_2) - G(u_1)G(u_2)] + o\left(\frac{1}{n^2}\right) \end{aligned}$$

and

$$\begin{aligned} (3.1.25) \quad & \sum_i \sum_{i's} \sum_{s'} \left(\frac{1}{n} - G(u_1)\right) \left(\frac{1'}{n} - G(u_2)\right) p_{i-1, s-1}^{(n-1)}(u_1, v_1) p_{1'-1, s'-1}^{(n-1)}(u_2, v_2) \\ &= \sum_{i=1}^n \left(\frac{1}{n} - G(u_1)\right) p_{i-1}^{(n-1)}(u_1) \cdot \sum_{i'=1}^n \left(\frac{1'}{n} - G(u_2)\right) p_{i'-1}^{(n-1)}(u_2) \\ &= \frac{(1-G(u_1))(1-G(u_2))}{n^2} \\ &= o\left(\frac{1}{n^2}\right). \end{aligned}$$

We see that the sum of the terms involving $J_1'(G(u_1), F(v_1)) J_1'(G(u_2), F(v_2))$ reduces to

$$\begin{aligned} (3.1.26) \quad & \textcircled{K} = \frac{(n-1)(n-2)}{n^2} \iiint J_1'(G(u_1), F(v_1)) J_1'(G(u_2), F(v_2)) \\ & \quad \times [G(u_1 \wedge u_2) - G(u_1)G(u_2)] v_1 v_2 dH(u_1, v_1) dH(u_2, v_2) \\ & \quad + o\left(\frac{1}{n}\right). \end{aligned}$$

Similarly the sum of the terms involving $J_2'(G(u_1), F(v_1))J_2'(G(u_2), F(v_2))$ by use of relations of the type (3.1.23) to (3.1.25) with $(\frac{1}{n} - G(u_1))(\frac{1}{n} - G(u_2))$ replaced by $(\frac{s}{n} - F(v_1))(\frac{s'}{n} - F(v_2))$, reduces to

$$(3.1.27) \quad \textcircled{L} = \frac{(n-1)(n-2)}{n^2} \iiint J_2'(G(u_1), F(v_1))J_2'(G(u_2), F(v_2)) \\ \times [F(v_1 \wedge v_2) - F(v_1)F(v_2)] v_1 v_2 dH(u_1, v_1) dH(u_2, v_2) \\ + O(1/n).$$

Further, we have the following relations (3.1.28) to (3.1.30) which are similar to those of (3.1.23) to (3.1.25):

$$(3.1.28) \quad \sum_{i < i'} \sum_{s \leq s'} \left(\frac{1}{n} - G(u_1) \right) \left(\frac{s'}{n} - F(v_2) \right) p_{i s i' s'}^{11}(u_1, v_1, u_2, v_2) \\ = \sum_{i=1}^{n-1} \sum_{s'=2}^n \left(\frac{1}{n} - G(u_1) \right) \left(\frac{s'}{n} - F(v_2) \right) p_{i s'}^{11}(u_1, v_2) \\ = \frac{1}{n^2} \sum_{i=1}^{n-1} \sum_{s'=2}^n [(i-1) - (n-2)G(u_1)] [(s'-2) - (n-2)F(v_2)] p_{i s'}^{11}(u_1, v_2) \\ + \frac{(1-2G(u_1))(2-2F(v_2))}{n^2} \\ = \frac{1}{n^2} \text{Cov} \left(\sum_{i=1}^{n-2} I(X_i \leq u_1), \sum_{j=1}^{n-2} I(Y_j \leq v_2) \right) + O\left(\frac{1}{n^2}\right) \\ = \frac{n-2}{n^2} [H(u_1, v_2) - G(u_1)F(v_2)] + O\left(\frac{1}{n^2}\right),$$

and similarly

$$(3.1.29) \quad \sum_{i < i'} \sum_{s > s'} \left(\frac{1}{n} - G(u_1) \right) \left(\frac{s'}{n} - F(v_2) \right) p_{i s i' s'}^{12}(u_1, v_1, u_2, v_2) \\ = \frac{n-2}{n^2} [H(u_1, v_2) - G(u_1)F(v_2)] + O\left(\frac{1}{n^2}\right),$$

$$\sum_{i > i'} \sum_{s < s'} \left(\frac{1}{n} - G(u_1) \right) \left(\frac{s'}{n} - F(v_2) \right) p_{i s i' s'}^{21}(u_1, v_1, u_2, v_2) \\ = \frac{n-2}{n^2} [H(u_1, v_2) - G(u_1)F(v_2)] + O\left(\frac{1}{n^2}\right),$$

$$\sum_{i > i'} \sum_{s > s'} \left(\frac{1}{n} - G(u_1) \right) \left(\frac{s'}{n} - F(v_2) \right) p_{i s i' s'}^{22}(u_1, v_1, u_2, v_2) \\ = \frac{n-2}{n^2} [H(u_1, v_2) - G(u_1)F(v_2)] + O\left(\frac{1}{n^2}\right),$$

and

$$(3.1.30) \quad \sum_i \sum_{i'} \sum_s \sum_{s'} \left(\frac{1}{n} - G(u_1) \right) \left(\frac{s'}{n} - F(v_2) \right) p_{i-1, s-1}^{(n-1)}(u_1, v_1) p_{i'-1, s'-1}^{(n-1)}(u_2, v_2) \\ = \sum_{i=1}^n \left(\frac{1}{n} - G(u_1) \right) p_{i-1}^{(n-1)}(u_1) \sum_{s'=1}^n \left(\frac{s'}{n} - F(v_2) \right) p_{s'-1}^{(n-1)}(v_2) \\ = \frac{(1-G(u_1))(1-F(v_2))}{n^2} \\ = O\left(\frac{1}{n^2}\right).$$

Now using the relations (3.1.28) to (3.1.30) and the similar kind which we

get when $\left(\frac{1}{n} - G(u_1) \right) \left(\frac{s'}{n} - F(v_2) \right)$ there is replaced by

$\left(\frac{1'}{n} - G(u_2) \right) \left(\frac{s}{n} - F(v_1) \right)$, the sum of terms involving

$J_1'(G(u_1), F(v_1)) J_2'(G(u_2), F(v_2))$ and $J_1'(G(u_2), F(v_2)) J_2'(G(u_1), F(v_2))$

reduces to

$$(3.1.31) \quad \textcircled{M} = \frac{2(n-1)(n-2)}{n^2} \iiint J_1'(G(u_1), F(v_1)) J_2'(G(u_2), F(v_2)) \\ \times [H(u_1, v_2) - G(u_1)F(v_2)] dH(u_1, v_1) dH(u_2, v_2) \\ + O(1/n).$$

By similar arguments as above we can see that all other terms will be of order $O(n^{-1/2})$ or higher. Since the arguments involved are essentially the same we shall only deal with two of those terms here, namely,

$J_1''(G(u_1), F(v_1))J(G(u_2), F(v_2))$ term and $J_1''(G(u_1), F(v_1))J_2'(G(u_2), F(v_2))$ term:

$$\begin{aligned} \text{Since} \quad & \sum_{i < i', s < s'} \left(\frac{1}{n} - G(u_1)\right)^2 p_{isi's'}^{11}(u_1, v_1, u_2, v_2) \\ &= \sum_{i < i', s < s'} \left(\frac{1}{n} - G(u_1)\right)^2 p_{isi's'}^{12}(u_1, v_1, u_2, v_2) \\ &= \sum_{i=1}^{n-1} \left(\frac{1}{n} - G(u_1)\right)^2 p_{i-1}^{(n-2)}(u_1) \\ &= \frac{1}{n^2} \sum_{i=1}^{n-1} [(1-i) - (n-2)G(u_1)]^2 p_{i-1}^{(n-2)}(u_1) + \frac{(1-2G(u_1))^2}{n^2} \\ &= \frac{(n-2)G(u_1)(1-G(u_1))}{n^2} + O\left(\frac{1}{n^2}\right), \end{aligned}$$

and similarly

$$\begin{aligned} & \sum_{i > i', s < s'} \left(\frac{1}{n} - G(u_1)\right)^2 p_{isi's'}^{21}(u_1, v_1, u_2, v_2) \\ &= \sum_{i > i', s < s'} \left(\frac{1}{n} - G(u_1)\right)^2 p_{isi's'}^{22}(u_1, v_1, u_2, v_2) \\ &= \sum_{i=2}^n \left(\frac{1}{n} - G(u_1)\right)^2 p_{i-2}^{(n-2)}(u_1) \\ &= \frac{(n-2)G(u_1)(1-G(u_1))}{n^2} + O\left(\frac{1}{n^2}\right), \end{aligned}$$

and

$$\begin{aligned}
 & \sum_{i=1}^n \sum_{i' < i} \sum_{s < s'} \left(\frac{1}{n} - G(u_1) \right)^2 p_{i-1, s-1}^{(n-2)}(u_1, v_1) p_{i'-1, s'-1}^{(n-2)}(u_2, v_2) \\
 &= \sum_{i=1}^n \left(\frac{1}{n} - G(u_1) \right)^2 p_{i-1}^{(n-1)}(u_1) \\
 &= \frac{(n-1)G(u_1)(1-G(u_1))}{n^2} + o\left(\frac{1}{n^2}\right),
 \end{aligned}$$

the sum of the terms involving $J_1''(G(u_1), F(v_1))J(G(u_2), F(v_2))$ reduces to

$$\begin{aligned}
 & \frac{(n-1)(n-2)}{n^2} \iiint J_1''(G(u_1), F(v_1))J(G(u_2), F(v_2))G(u_1)(1-G(u_1))v_1v_2 \\
 & \quad dH(u_1, v_1)dH(u_2, v_2) \\
 & - \frac{n-1}{n} \iiint J_1''(G(u_1), F(v_1))J(G(u_2), F(v_2))G(u_1)(1-G(u_1))v_1v_2 \\
 & \quad dH(u_1, v_1)dH(u_2, v_2) \\
 & + o\left(\frac{1}{n}\right) \\
 & = o\left(\frac{1}{n}\right).
 \end{aligned}$$

The term involving $J_1''(G(u_1), F(v_1))J_2'(G(u_2), F(v_2))$ from the second integral on the R.H.S. of (3.1.13) is

$$\begin{aligned}
 I &= (n-1) \iiint \left[\sum_{i < i'} \sum_{s < s'} \left(\frac{1}{n} - G(u_1) \right)^2 \left(\frac{s'}{n} - F(v_2) \right) p_{is, i's'}^{11}(u_1, v_1, u_2, v_2) \right] \\
 & \quad J_1''(G(u_1), F(v_1))J_2'(G(u_2), F(v_2))v_1v_2 dH(u_1, v_1)dH(u_2, v_2).
 \end{aligned}$$

Write

$$\begin{aligned}
 (3.1.32) \quad P &= \sum_{i < i'} \sum_{s < s'} \left(\frac{1}{n} - G(u_1) \right)^2 \left(\frac{s'}{n} - F(v_2) \right) p_{is, i's'}^{11}(u_1, v_1, u_2, v_2) \\
 &= \sum_{i=1}^{n-1} \sum_{s'=2}^n \left(\frac{1}{n} - G(u_1) \right)^2 \left(\frac{s'}{n} - F(v_2) \right) p_{is, i's'}^{11}(u_1, v_2)
 \end{aligned}$$

$$= \frac{1}{n^3} \sum_{i=1}^{n-1} \sum_{s'=2}^n [(i-1)-(n-2)G(u_1) + (1-2G(u_1))]^2 \\ [(s'-2)-(n-2)F(v_2) + (2-2F(v_2))] p_{1s'}^{11}(u_1, v_2).$$

Now if $T_1 = \sum_{i=1}^{n-2} I(X_i \leq u_1)$ and $T_2 = \sum_{j=1}^{n-2} I(Y_j \leq v_2)$, then T_1 is Binomial $(n-2, G(u_1))$, T_2 is Binomial $(n-2, F(v_2))$ and we have the relations

$$(3.1.33) \quad \sum_{i=1}^{n-1} [(i-1)-(n-2)G(u_1)] p_{i-1}^{(n-2)}(u_1) = E(T_1 - ET_1) = 0,$$

$$\sum_{s'=2}^n [(s'-2)-(n-2)F(v_2)] p_{s'-2}^{(n-2)}(v_2) = E(T_2 - ET_2) = 0,$$

$$\sum_{i=1}^{n-1} [(i-1)-(n-2)G(u_1)]^2 p_{i-1}^{(n-2)}(u_1) = \text{Var}(T_1) = (n-2)G(u_1)(1-G(u_1)),$$

$$\sum_{s'=2}^n [(s'-2)-(n-2)F(v_2)]^2 p_{s'-2}^{(n-2)}(v_2) = \text{Var}(T_2) = (n-2)F(v_2)(1-F(v_2)),$$

$$\sum_{i=1}^{n-1} \sum_{s'=2}^n [(i-1)-(n-2)G(u_1)][(s'-2)-(n-2)F(v_2)] p_{i-1, s'-2}^{11}(u_1, v_2) \\ = (n-2)[H(u_1, v_2) - G(u_1)F(v_2)],$$

$$\text{and} \quad \left| \sum_{i=1}^{n-1} \sum_{s'=2}^n [(i-1)-(n-2)G(u_1)]^2 [(s'-2)-(n-2)F(v_2)] p_{i-1, s'-2}^{11}(u_1, v_2) \right| \\ = |E[(T_1 - ET_1)^2 (T_2 - ET_2)]| \\ \leq [E(T_1 - ET_1)^4 E(T_2 - ET_2)^2]^{1/2} \\ = [O(n^2)O(n)]^{1/2} = O(n^{3/2}).$$

Using the relations (3.1.16) and (3.1.33) in (3.1.32) we see that

$$\begin{aligned}
 P &= \frac{1}{n^3} \sum_{i=1}^{n-1} \sum_{s'=2}^n [(i-1)-(n-2)G(u_1)]^2 [(s'-2)-(n-2)F(v_2)] p_{1s'}^{11}(u_1, v_2) \\
 &+ \frac{(2-2F(v_2))}{n^3} \sum_{i=1}^n [(i-1)-(n-2)G(u_1)]^2 p_{1-1}^{(n-2)}(u_1) \\
 &+ \frac{2(1-2G(u_1))}{n^3} \sum_{i=1}^{n-1} \sum_{s'=2}^n [(i-1)-(n-2)G(u_1)] [(s'-2)-(n-2)F(v_2)] p_{1s'}^{11}(u_1, v_2) \\
 &+ \frac{2(1-2G(u_1))^2(2-2F(v_2))}{n^3} \\
 &= O(n^{-3/2}) + O(n^{-2}) + O(n^{-2}) + O(n^{-3})
 \end{aligned}$$

and consequently $I = O(n^{-1/2})$ since J_1'', J_2' are bounded and $E Y^2 < \infty$. On the same lines it is easy to see that the term involving

$J_1''(G(u_1), F(v_1)) J_2'(G(u_2), F(v_2))$ from every integral on the R.H.S. of (3.1.13)

is $O(n^{-1/2})$.

All the other terms involving the higher order derivatives of J can be handled exactly as the $J_1''(G(u_1), F(v_1)) J_2'(G(u_2), F(v_2))$ term and can be shown to be asymptotically negligible. Hence (3.1.17), (3.1.18), (3.1.20) to (3.1.22), (3.1.26) to (3.1.27) and (3.1.31) together yields

$$\begin{aligned}
 (3.1.34) \quad n \text{Var}(S_n) &= \iint J^2(G(u), F(v)) v^2 dH(u, v) \\
 &- \iiint J(G(u_1), F(v_1)) J(G(u_2), F(v_2)) v_1 v_2 dH(u_1, v_1) dH(u_2, v_2) \\
 &+ \frac{2(n-1)}{n} \iiint J(G(u_1), F(v_1)) J_1'(G(u_2), F(v_2)) \\
 &\quad \times [I(u_1 < u_2) - G(u_2)] v_1 v_2 dH(u_1, v_1) dH(u_2, v_2)
 \end{aligned}$$

continued

$$\begin{aligned}
& + \frac{2(n-1)}{n} \iiint J(G(u_1), F(v_1)) J_2'(G(u_2), F(v_2)) \\
& \quad \times [I(v_1, v_2) - F(v_2)] v_1 v_2 dH(u_1, v_1) dH(u_2, v_2) \\
& + \frac{(n-1)(n-2)}{n^2} \iiint J_1'(G(u_1), F(v_1)) J_1'(G(u_2), F(v_2)) \\
& \quad \times [G(u_1, u_2) - G(u_1)G(u_2)] v_1 v_2 dH(u_1, v_1) dH(u_2, v_2) \\
& + \frac{(n-1)(n-2)}{n^2} \iiint J_2'(G(u_1), F(v_1)) J_2'(G(u_2), F(v_2)) \\
& \quad \times [F(v_1, v_2) - F(v_1)F(v_2)] v_1 v_2 dH(u_1, v_1) dH(u_2, v_2) \\
& + \frac{2(n-1)(n-2)}{n^2} \iiint J_1'(G(u_1), F(v_1)) J_2'(G(u_2), F(v_2)) \\
& \quad \times [H(u_1, v_2) - G(u_1)F(v_2)] v_1 v_2 dH(u_1, v_1) dH(u_2, v_2) \\
& + o(n^{-1/2}).
\end{aligned}$$

The Lemma now follows by taking limit, as $n \rightarrow \infty$, in (3.1.34). $\square$

From lemma 2.3.2, lemma 3.1.1 and equation (2.1.3) it now follows that

$$\lim_{n \rightarrow \infty} E(T_n - \hat{T}_n)^2 = \lim_{n \rightarrow \infty} [\text{Var}(T_n) - \text{Var}(\hat{T}_n)] = 0$$

and hence T_n and $\hat{T}_n$ are asymptotically equivalent in mean square.

3.2 ASYMPTOTIC NORMALITY OF S_n . In this section we shall prove Theorem 3.1.1 which establishes the asymptotic normality of S_n .

PROOF OF THEOREM 3.1.1. Since $T_n = n^{1/2}(S_n - ES_n)$ and $\hat{T}_n = n^{1/2}(\hat{S}_n - E\hat{S}_n)$ are asymptotically equivalent in mean square, it suffices to prove that $\hat{T}_n$ is

asymptotically normally distributed (e.g. see Corollary 2, page 349 of [10]). Using (2.2.2) and the notation set out in (2.3.1), we can write $\hat{S}_n$ (see (2.3.2)) and, therefore $\hat{T}_n$ as

$$(3.2.1) \quad \hat{T}_n = n^{1/2}(\hat{S}_n - E\hat{S}_n) = n^{-1/2} \sum_{k=1}^n Z_{kn}$$

where for $k = 1, 2, \dots, n$,

$$\begin{aligned} Z_{kn} = & Y_k \tilde{Q}_n(X_k, Y_k) - \iint \tilde{Q}_n(u, v) v dH(u, v) \\ & + \iint (n-1) [Q_n(u, v) - Q_n^*(u, v)] [I(X_k \leq u) - G(u)] v dH(u, v) \\ & + \iint (n-1) [\tilde{Q}_n(u, v) - Q_n(u, v) + Q_n^*(u, v) - \tilde{Q}_n^*(u, v)] \\ & \quad \times [I(X_k \leq u, Y_k \leq v) - H(u, v)] v dH(u, v) \\ & + \iint (n-1) [\tilde{Q}_n^*(u, v) - Q_n^*(u, v)] [I(Y_k \leq v) - F(v)] v dH(u, v). \end{aligned}$$

Now define for $k = 1, 2, \dots, n$,

$$\begin{aligned} Z_k = & Y_k J(G(X_k), F(Y_k)) - \iint J(G(u), F(v)) v dH(u, v) \\ & + \iint J_1'(G(u), F(v)) [I(X_k \leq u) - G(u)] v dH(u, v) \\ & + \iint J_2'(G(u), F(v)) [I(Y_k \leq v) - F(v)] v dH(u, v). \end{aligned}$$

Then

$$\begin{aligned} (3.2.2) \quad \text{Var}(n^{-1/2} \sum_{k=1}^n Z_{kn} - n^{-1/2} \sum_{k=1}^n Z_k) &= \text{Var}(Z_{1n} - Z_1) \\ &= \text{Var}[Y_1 [\tilde{Q}_n(X_1, Y_1) - J(G(X_1), F(Y_1))]] \\ &\quad - \iint [\tilde{Q}_n(u, v) - J(G(u), F(v))] v dH(u, v) \end{aligned}$$

continued

$$\begin{aligned}
& + \iint [(n-1)(Q_n(u,v) - Q_n^*(u,v)) - J_1'(G(u), F(v))] \\
& \quad \times [I(X_1 \leq u) - G(u)] v dH(u,v) \\
& + \iint (n-1)[\bar{Q}_n(u,v) - Q_n(u,v) + Q_n^*(u,v) - \bar{Q}_n^*(u,v)] \\
& \quad \times [I(X_1 \leq u, Y_1 \leq v) - H(u,v)] v dH(u,v) \\
& + \iint [(n-1)(\bar{Q}_n^*(u,v) - Q_n^*(u,v)) - J_2'(G(u), F(v))] \\
& \quad \times [I(Y_1 \leq v) - F(v)] v dH(u,v).
\end{aligned}$$

It follows from arguments similar to those used in the proof of Lemma 2.3.2 (see (2.3.5)) that the right hand side of (3.2.2) goes to 0 as $n \rightarrow \infty$,

which implies that $n^{-1/2} \sum_{k=1}^n Z_{kn}$ and $n^{-1/2} \sum_{k=1}^n Z_k$ are asymptotically equivalent in mean square. But the Z_k 's are i.i.d. with mean 0 and finite variance $\sigma^2(J, H)$. So by the standard central limit theorem $n^{-1/2} \sum_{k=1}^n Z_k$ is asymptotically $N(0, \sigma^2(J, H))$ and the proof of Theorem 3.1.1 is complete. $\square$

PROOF OF COROLLARY 3.1.1. From (2.1.1) and (3.1.6) we have

$$\begin{aligned}
(3.2.3) \quad E(S_n) &= n^{-1} \sum_{i=1}^n E \left[J\left(\frac{1}{n}, \frac{R_{in}}{n}\right) Y_{[i:n]} \right] \\
&= \iint v \left[\sum_{i=1}^n \sum_{s=1}^n J\left(\frac{1}{n}, \frac{s}{n}\right) p_{i-1, s-1}^{(n-1)}(u, v) \right] dH(u, v)
\end{aligned}$$

where $p_{i,s}^{(n)}(u, v)$ is as defined by (2.2.3). Taylor's expansion of $J(\frac{1}{n}, \frac{s}{n})$ about $(G(u), F(v))$ yields

$$\begin{aligned}
E(S_n) = & \iint \sum_{i=1}^n \sum_{s=1}^n [J(G(u), F(v)) + (\frac{1}{n} - G(u))J_1'(G(u), F(v)) \\
& + (\frac{s}{n} - F(v))J_2'(G(u), F(v)) + (\frac{1}{n} - G(u))^2 J_1''(\Delta_{1n}, \Delta_{sn}) \\
& + (\frac{s}{n} - F(v))^2 J_2''(\Delta_{1n}, \Delta_{sn}) + (\frac{1}{n} - G(u))(\frac{s}{n} - F(v)) \\
& J_{12}''(\Delta_{1n}, \Delta_{sn})] p_{1-1, s-1}^{(n-1)}(u, v) dH(u, v)
\end{aligned}$$

where Δ_{1n} lies between $\frac{1}{n}$ and $G(u)$ and Δ_{sn} lies between $\frac{s}{n}$ and $F(v)$.

On using the appropriate relations from (3.1.33), and since J_1'' , J_2'' , J_{12}'' are bounded, $E |Y| < \infty$, we see that

$$\begin{aligned}
(3.2.4) \quad E(S_n) &= \iint J(G(u), F(v)) dH(u, v) + O(\frac{1}{n}) \\
&= \mu(J, H) + O(\frac{1}{n})
\end{aligned}$$

and it follows that

$$(3.2.5) \quad n^{1/2} (E(S_n) - \mu(J, H)) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad \square$$

Now in view of (3.2.5), $E(S_n)$ can be replaced by $\mu(J, H)$ in Theorem 3.1.1.

3.3. TWO IMPORTANT SPECIAL CASES. In this section we shall prove Theorem 3.1.2, which covers the results of Stigler [17] and Yang [22] as special cases of Theorem 3.1.1.

PROOF OF THEOREM 3.1.2.

First observe that our statistic S_n itself reduces to that of Stigler (1974) and Yang (1981) under the assumptions $J(x, y) = J(y)$, $0 \leq x, y \leq 1$, and $J(x, y) = J(x)$, $0 \leq x, y \leq 1$, respectively.

For convenience we will write $V(S_n)$, $V_S(S_n)$ and $V_Y(S_n)$ to denote the expressions for the asymptotic variance of the statistic $n^{1/2}S_n$ as considered in this thesis, in Stigler [17] and in Yang [22] respectively. Also let $\sigma^2(x) = \text{Var}(Y|X=x)$, $m(x) = E(Y|X=x)$.

We will first prove part (b): Under the assumption $J(x,y) = J(x)$, $0 \leq x, y \leq 1$, $V(S_n)$ given by (2.3.9) reduces to

$$\begin{aligned}
 (3.3.1) \quad V(S_n) &= \iint J^2(G(u)) v^2 dH(u,v) - \iiint J(G(u_1)) J(G(u_2)) v_1 v_2 dH(u_1, v_1) \\
 &\quad dH(u_2, v_2) \\
 &\quad + 2 \iiint J(G(u_1)) J'(G(u_2)) [I(u_1 \leq u_2) - G(u_2)] v_1 v_2 dH(u_1, v_1) \\
 &\quad dH(u_2, v_2) \\
 &\quad + \iiint J'(G(u_1)) J'(G(u_2)) [G(u_1 \wedge u_2) - G(u_1) G(u_2)] v_1 v_2 \\
 &\quad dH(u_1, v_1) dH(u_2, v_2) \\
 &= \int J^2(G(u)) (\sigma^2(u) + m^2(u)) dG(u) \\
 &\quad - \iint J(G(u_1)) J(G(u_2)) m(u_1) m(u_2) dG(u_1) dG(u_2) \\
 &\quad + 2 \iint J(G(u_1)) J'(G(u_2)) [I(u_1 \leq u_2) - G(u_2)] m(u_1) m(u_2) \\
 &\quad dG(u_1) dG(u_2) \\
 &\quad + \iint J'(G(u_1)) J'(G(u_2)) [G(u_1 \wedge u_2) - G(u_1) G(u_2)] m(u_1) m(u_2) \\
 &\quad dG(u_1) dG(u_2).
 \end{aligned}$$

Now from Yang [22], we have

$$\begin{aligned}
 (3.3.2) \quad V_Y(S_n) &= \int J^2(G(u)) \sigma^2(u) dG(u) \\
 &\quad + \iint [G(u_1 \wedge u_2) - G(u_1) G(u_2)] \times J(G(u_1)) J(G(u_2)) dm(u_1) dm(u_2).
 \end{aligned}$$

$$\begin{aligned}
&= \int m(u_2) J^2(G(u_2)) G(u_2) (1-G(u_2)) dm(u_2) \\
&\quad + \int J^2(G(u_1)) m^2(u_1) (1-G(u_1)) dG(u_1) \\
&\quad + \int J(G(u_1)) J'(G(u_1)) G(u_1) (1-G(u_1)) m^2(u_1) dG(u_1) \\
&\quad - \int \int_{u_1 \leq u_2} J(G(u_1)) J(G(u_2)) m(u_1) m(u_2) dG(u_1) dG(u_2) \\
&\quad - \int \int_{u_1 \leq u_2} J(G(u_2)) J'(G(u_1)) G(u_1) m(u_1) m(u_2) dG(u_1) dG(u_2) \\
&\quad + \int \int_{u_1 \leq u_2} J(G(u_1)) J'(G(u_2)) (1-G(u_2)) m(u_1) m(u_2) dG(u_1) dG(u_2) \\
&\quad + \int \int_{u_1 \leq u_2} J'(G(u_1)) J'(G(u_2)) G(u_1) (1-G(u_2)) m(u_1) m(u_2) dG(u_1) dG(u_2).
\end{aligned}$$

Proceeding exactly on the same lines we can see that

$$\begin{aligned}
(3.3.5) \quad I_2 &= - \int m(u_2) J^2(G(u_2)) G(u_2) (1-G(u_2)) dm(u_2) \\
&\quad + \int J^2(G(u_1)) m^2(u_1) G(u_1) dG(u_1) \\
&\quad - \int J(G(u_1)) J'(G(u_1)) m^2(u_1) G(u_1) (1-G(u_1)) dG(u_1) \\
&\quad - \int \int_{u_1 > u_2} J(G(u_1)) J(G(u_2)) m(u_1) m(u_2) dG(u_1) dG(u_2) \\
&\quad - \int \int_{u_1 > u_2} J(G(u_1)) J'(G(u_2)) G(u_2) m(u_1) m(u_2) dG(u_1) dG(u_2) \\
&\quad + \int \int_{u_1 > u_2} J(G(u_2)) J'(G(u_1)) (1-G(u_1)) m(u_1) m(u_2) dG(u_1) dG(u_2) \\
&\quad + \int \int_{u_1 > u_2} J'(G(u_1)) J'(G(u_2)) G(u_2) (1-G(u_1)) m(u_1) m(u_2) dG(u_1) dG(u_2).
\end{aligned}$$

Now using (3.3.3) to (3.3.5) in (3.3.2) we see that

$$V(S_n) = V_Y(S_n)$$

where $V(S_n)$ is as given by (3.3.1).

(a) First observe that under the assumption $J(x,y) = J(y)$, $0 \leq x \leq y \leq 1$

$V(S_n)$ given by (2.3.9) reduces to

$$\begin{aligned} (3.3.6) \quad V(S_n) = & \int J^2(F(v))v^2 dF(v) - \iint J(F(v_1))J(F(v_2))v_1v_2 dF(v_1)dF(v_2) \\ & + 2 \iint J(F(v_1))J'(F(v_2))[I(v_1 \leq v_2) - F(v_2)]v_1v_2 dF(v_1)dF(v_2) \\ & + \iint J'(F(v_1))J'(F(v_2))[F(v_1 \wedge v_2) - F(v_1)F(v_2)]v_1v_2 dF(v_1)dF(v_2), \end{aligned}$$

and from Stigler [17]

$$(3.3.7) \quad V_S(S_n) = \iint J(F(v_1))J(F(v_2))[F(v_1 \wedge v_2) - F(v_1)F(v_2)]du_1 dv_1.$$

Now by writing $m(u) = u$ and replacing G by F in (3.3.3) the expression on the L.H.S. of (3.3.3) reduces to that of (3.3.7) and consequently it is easy to see that equations (3.3.3) to (3.3.5) with the corresponding changes yields

$$V(S_n) = V_S(S_n).$$

□

CHAPTER IV

A CONSISTENT CONDITIONAL QUANTILE ESTIMATOR

This chapter essentially deals with the consistency of a nonparametric estimator $Q_{n,x_0}(\lambda)$, of the λ^{th} quantile $0 < \lambda < 1$, proposed by Mehra [12] (See 4.1.1 below) of the conditional distribution of Y given $X = x_0$. Certain preliminary results obtained in this regard are also referred to in Chapter V. In section 4.1 the main result of this chapter, namely, the consistency of the estimator $Q_{n,x_0}(\lambda)$ is stated along with a brief outline of the method of proof and a useful decomposition of $Q_{n,x_0}(\lambda)$ is obtained which is basic for all the subsequent results. In section 4.2 we define the conditional empirical process and show that its supremum is stochastically bounded under certain conditions. Though this result is required here only in proving our main theorem, it may also be of some independent interest. Sections 4.3 and 4.4 are devoted to prove some further preliminary results and the proof of the main result is given in Section 4.5.

4.1 THE ESTIMATOR $Q_{n,x_0}(\lambda)$. Let $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$ be independent and identically distributed rv's with absolutely continuous cumulative distribution function (cdf) $H(x, y)$, and continuous joint probability density function $h(x, y)$, $-\infty < x, y < \infty$. Let $X_{(1:n)} \leq X_{(2:n)} \leq \dots \leq X_{(n:n)}$ denote the X -variate order statistics and $Y_{[1:n]}, Y_{[2:n]}, \dots, Y_{[n:n]}$ denote the corresponding Y -variate induced order statistics. Let $\{a_n\}$ be a sequence of positive numbers such that $a_n \rightarrow 0$ as $n \rightarrow \infty$. The following estimator $Q_{n,x_0}(\lambda)$ is proposed by Mehra [12] as a

nonparametric kernel estimator of the λ^{th} quantile of the conditional distribution of Y given $X = x_0$ (for $-\infty < x_0 < \infty$, $0 < \lambda < 1$):

$$\begin{aligned}
 (4.1.1.) \quad Q_{n,x_0}(\lambda) &= \frac{1}{na_n^2} \sum_{i=1}^n K\left(\frac{G_n(X_{(i:n)}) - G_n(x_0)}{a_n}, \frac{F_{n,x_0}(Y_{[i:n]}) - \lambda}{a_n}\right) Y_{[i:n]} \\
 &= \frac{1}{na_n^2} \sum_{i=1}^n K\left(\frac{G_n(X_i) - G_n(x_0)}{a_n}, \frac{F_{n,x_0}(Y_i) - \lambda}{a_n}\right) Y_i,
 \end{aligned}$$

where $G_n(\cdot)$ is the marginal empirical distribution function of X -observations,

$F_{n,x_0}(\cdot)$ is the "conditional empirical distribution function" of

Y -observations given $x = x_0$ defined by

$$(4.1.2) \quad F_{n,x_0}(t) = \frac{1}{na_n} \sum_{j=1}^n K^*\left(\frac{G_n(X_j) - G_n(x_0)}{a_n}\right) I(Y_j \leq t),$$

and $K^*(\cdot)$, $K(\cdot, \cdot)$ are, respectively, a univariate and bivariate kernel density functions satisfying the following conditions:

(4.1.3) $K(\cdot, \cdot)$ is a bivariate continuous probability density function (pdf) with $v(4 \leq v \leq 6)$ bounded partial derivatives and has compact support,

(4.1.4) $K^*(\cdot)$ is a univariate continuous probability density function with $v^*(2 \leq v^* \leq 4)$ bounded derivatives and has compact support,

$$(4.1.5) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u k(u, v) du dv = 0 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v k(u, v) du dv,$$

$$(4.1.6.) \quad \int_{-\infty}^{\infty} t K^*(t) dt = 0.$$

Let $G, g(F, f)$ denote, respectively, the marginal cdf and pdf of X (of Y). Also, let $F_x(f_x)$ denote the conditional cdf(pdf) of Y given $X = x$. The λ^{th} quantile of the conditional distribution of Y given $X = x_0$ is defined by

$$(4.1.7) \quad Q_{x_0}(\lambda) = F_{x_0}^{-1}(\lambda) = \inf\{t: F_{x_0}(t) \geq \lambda\}, \quad 0 < \lambda < 1.$$

The main object of this chapter is to prove the consistency of the estimator $Q_{n, x_0}(\lambda)$ viz. to prove that $Q_{n, x_0}(\lambda) \xrightarrow{P} Q_{x_0}(\lambda)$ as $n \rightarrow \infty$. Our method of proof is as follows: First, we shall decompose $Q_{n, x_0}(\lambda)$, by the Taylor expansion using the differentiability properties of K , as

$$(4.1.8) \quad Q_{n, x_0}(\lambda) = I_1 + I_2 + I_3 + R_n$$

where I_1 is the "constant" term, I_2 and I_3 are the first order derivative terms and R_n includes all the terms with second and higher order derivatives of K . Using the differentiability properties of K^* , the I_3 term is further decomposed by the Taylor expansion to $I_{31} + I_{32} - I_{30} + R'_n$, where I_{31} and I_{32} are, respectively, the "constant" and first order derivative terms with respect to K^* , I_{30} is that part of I_3 term which does not include K^* and R'_n includes all the terms with second and higher order derivatives of K^* , so that we have the final decomposition of $Q_{n, x_0}(\lambda)$

given by

$$(4.1.9) \quad \begin{aligned} Q_{n, x_0}(\lambda) &= I_1 + I_2 + I_{31} + I_{32} - I_{30} + R_n + R'_n \\ &= S_{n, x_0}(\lambda) + R_n + R'_n. \end{aligned}$$

(For the exact expressions of $I_1, I_2, I_{31}, I_{32}, I_{30}, R_n$ and R'_n (see (4.1.20) to (4.1.24)) . From (4.1.9) we write

$$(4.1.10) \quad Q_{n,x_0}(\lambda) - Q_{x_0}(\lambda) = [S_{n,x_0}(\lambda) - Q_{x_0}(\lambda)] + R_n + R'_n \text{ and then show}$$

that each of the terms R_n, R'_n and $[S_{n,x_0}(\lambda) - Q_{x_0}(\lambda)]$ converge to zero in probability as $n \rightarrow \infty$, which proves the consistency of $Q_{n,x_0}(\lambda)$. Convergence of $[S_{n,x_0}(\lambda) - Q_{x_0}(\lambda)]$ to zero in probability is proved by showing that $na_n^2 \text{Var}(S_{n,x_0}(\lambda) - Q_{x_0}(\lambda))$ goes to a finite value as $n \rightarrow \infty$ and that $E(S_{n,x_0}(\lambda) - Q_{x_0}(\lambda))$ goes to zero as $n \rightarrow \infty$. The result that $na_n^2 \text{Var}(S_{n,x_0}(\lambda))$ goes to a finite value as $n \rightarrow \infty$ is required in Chapter V * to prove the asymptotic normality of $Q_{n,x_0}(\lambda)$ in a properly normalized form.

We shall assume that $E(X^2) < \infty$ and $E(Y^2) < \infty$ throughout Chapters IV and V. We also make the following assumptions on the joint distribution function $H(x,y)$:

(4.1.11) $H(...)$ has bounded partial derivatives up to the fourth order in some neighborhood of $(x_0, F_{x_0}^{-1}(\lambda))$.

(4.1.12) $f_{x_0}(\cdot)$ has two bounded derivatives in some neighborhood of $F_{x_0}^{-1}(\lambda)$.

(4.1.13) $-h(...)$ is positive in some neighborhood of $(x_0, F_{x_0}^{-1}(\lambda))$.

(4.1.14) $f_x(y)$ is bounded uniformly in x and y , $-\infty < x < \infty$, $-\infty < y < \infty$.

(4.1.15) $H_1'(x,y)$, $H_1''(x,y)$, $H_1'''(x,y)$ and $H_1^{(4)}(x,y)$ exist for all $y \in \mathbb{R}$ and x in some neighborhood of x_0 and are uniformly bounded in y .

We now state the main result of this chapter.

THEOREM 4.1.1. Let $Q_{n,x_0}(\lambda)$, $0 < \lambda < 1$ be as defined by (4.1.1).

Suppose assumptions (4.1.3) to (4.1.6) on K and K^* are satisfied with $v = 4$ and $v^* = 3$ and assumptions (4.1.11) to (4.1.15) on the joint distribution H are also satisfied. Then for each sequence $\{a_n\}$ of positive numbers with $a_n \rightarrow 0$, $na_n^4 \rightarrow \infty$, but $na_n^5 \rightarrow 0$ we have

$$Q_{n,x_0}(\lambda) \rightarrow Q_{x_0}(\lambda) \text{ in probability, as } n \rightarrow \infty.$$

For the proof of the theorem, given in Section 4.5, we first obtain the representation (4.1.9) of $Q_{n,x_0}(\lambda)$, and prove some preliminary results given as lemmas 4.1.1, 4.2.1, 4.3.1 and 4.4.1 to 4.4.4.

Now observe that $Q_{n,x_0}(\lambda)$ defined by (4.1.1) may be written as

$$(4.1.16) \quad Q_{n,x_0}(\lambda) = a_n^{-2} \iint K\left(\frac{G_n(x_1) - G_n(x_0)}{a_n}, \frac{F_{n,x_0}(y_1) - \lambda}{a_n}\right) y_1 dH_n(x_1, y_1)$$

where H_n is the bivariate empirical distribution function of (X,Y) . Since K is four times differentiable, the Taylor expansion of K about

$$\left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n}\right) \text{ yields}$$

$$\begin{aligned}
(4.1.17) \quad Q_{n,x_0}(\lambda) &= a_n^{-2} \iint K \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 dH_n(x_1, y_1) \\
&+ a_n^{-3} \iint [G_n(x_1) - G_n(x_0) - G(x_1) + G(x_0)] \\
&\quad K'_1 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 dH_n(x_1, y_1) \\
&+ a_n^{-3} \iint [F_{n,x_0}(y_1) - F_{x_0}(y_1)] K'_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 dH_n(x_1, y_1) \\
&+ \sum_{m=2}^3 \frac{a_n^{-2-m}}{m!} \iint [(G_n(x_1) - G_n(x_0) - G(x_1) + G(x_0)) \frac{\partial}{\partial t_1} \\
&\quad + (F_{n,x_0}(y_1) - F_{x_0}(y_1)) \frac{\partial}{\partial t_2}]^m \\
&\quad K \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 dH_n(x_1, y_1) \\
&+ \frac{a_n^{-6}}{4!} \iint [G_n(x_1) - G_n(x_0) - G(x_1) + G(x_0)] \frac{\partial}{\partial t_1} \\
&\quad + (F_{n,x_0}(y_1) - F_{x_0}(y_1)) \frac{\partial}{\partial t_2}]^4 K(\Delta_{x_1,n}, \Delta_{y_1,n}) y_1 dH_n(x_1, y_1) \\
&= I_1 + I_2 + I_3 + \sum_{m=2}^4 \eta_{nm}
\end{aligned}$$

where $K'_1(\theta_1, \theta_2)$ is the first order partial derivative of K with respect to the

i th argument evaluated at (θ_1, θ_2) , $i = 1, 2$,

$\Delta_{x_1,n}$ lies between $a_n^{-1}[G_n(x_1) - G_n(x_0)]$ and $a_n^{-1}[G(x_1) - G(x_0)]$,

$\Delta_{y_1,n}$ lies between $a_n^{-1}[F_{n,x_0}(y_1) - \lambda]$ and $a_n^{-1}[F_{x_0}(y_1) - \lambda]$.

Similarly, since K^* is three times differentiable, by the Taylor expansion of K^* , $F_{n,x_0}(t)$ defined by (4.1.2) may be expressed as

$$\begin{aligned}
 (4.1.18) \quad F_{n,x_0}(t) = & a_n^{-1} \iint K^* \left(\frac{G(x_2) - G(x_0)}{a_n} \right) I(y_2 \leq t) dH_n(x_2, y_2) \\
 & + a_n^{-2} \iint [G_n(x_2) - G_n(x_0) - G(x_2) + G(x_0)] \\
 & \quad K^{*'} \left(\frac{G(x_2) - G(x_0)}{a_n} \right) I(y_2 \leq t) dH_n(x_2, y_2) \\
 & + \frac{a_n^{-3}}{2!} \iint [G_n(x_2) - G_n(x_0) - G(x_2) + G(x_0)]^2 \\
 & \quad K^{*''} \left(\frac{G(x_2) - G(x_0)}{a_n} \right) I(y_2 \leq t) dH_n(x_2, y_2) \\
 & + \frac{a_n^{-4}}{3!} \iint [G_n(x_2) - G_n(x_0) - G(x_2) + G(x_0)]^3 K^{*'''}(\Delta_{x_2,n}) \\
 & \quad I(y_2 \leq t) dH_n(x_2, y_2)
 \end{aligned}$$

where $K^{*'}, K^{*''}, K^{*'''}$ are, respectively, the first, second and third derivatives of K^* , and $\Delta_{x_2,n}$ lies between $a_n^{-1}[G_n(x_2) - G(x_0)]$ and $a_n^{-1}[G(x_2) - G(x_0)]$. In view of (4.1.18), I_3 of (4.1.17) can be written as

$$(4.1.19) \quad I_3 = a_n^{-4} \iiint K^* \left(\frac{G(x_2) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right)$$

$$y_1 I(y_2 \leq y_1) dH_n(x_1, y_1) dH_n(x_2, y_2)$$

$$+ a_n^{-5} \iiint [G_n(x_2) - G_n(x_0) - G(x_2) + G(x_0)]$$

$$K^* \left(\frac{G(x_2) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right)$$

$$y_1 I(y_2 \leq y_1) dH_n(x_1, y_1) dH_n(x_2, y_2)$$

$$+ \frac{a_n^{-6}}{2} \iiint [G_n(x_2) - G_n(x_0) - G(x_2) + G(x_0)]^2$$

$$K^{**} \left(\frac{G(x_2) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right)$$

$$y_1 I(y_2 \leq y_1) dH_n(x_1, y_1) dH_n(x_2, y_2)$$

$$+ \frac{a_n^{-7}}{3!} \iiint [G_n(x_2) - G_n(x_0) - G(x_2) + G(x_0)]^3$$

$$K^{***}(\Delta_{x_2, n}) K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right)$$

$$y_1 I(y_2 \leq y_1) dH_n(x_1, y_1) dH_n(x_2, y_2)$$

$$- a_n^{-3} \iint F_{x_0}(y_1) K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 dH_n(x_1, y_1)$$

$$= I_{31} + I_{32} + \sum_{m=2}^{\infty} \zeta_{nm} - I_{30}.$$

From (4.1.17) and (4.1.19) we now have the representation of $Q_{n,x_0}(\lambda)$ as

$$(4.1.20) \quad Q_{n,x_0}(\lambda) = I_1 + I_2 + I_{31} + I_{32} - I_{30} + \sum_{m=2}^3 \zeta_{nm} + \sum_{m=2}^4 \eta_{nm}$$

$$= S_{n,x_0}(\lambda) + \sum_{m=2}^3 \zeta_{nm} + \sum_{m=2}^4 \eta_{nm}$$

where $S_{n,x_0}(\lambda)$ is given by (4.1.21), ζ_{nm} are given by (4.1.22) to

(4.1.23) and η_{nm} are given by (4.1.24) to (4.1.25) below:

$$(4.1.21) \quad S_{n,x_0}(\lambda) = I_1 + I_2 + I_{31} + I_{32} - I_{30}$$

$$= a_n^{-2} \iint K\left(\frac{G(x_1)-G(x_0)}{a_n}, \frac{F_{x_0}(y_1)-\lambda}{a_n}\right) y_1 dH_n(x_1, y_1)$$

$$+ a_n^{-3} \iint [G_n(x_1)-G_n(x_0)-G(x_1)+G(x_0)]$$

$$K_1\left(\frac{G(x_1)-G(x_0)}{a_n}, \frac{F_{x_0}(y_1)-\lambda}{a_n}\right) y_1 dH_n(x_1, y_1)$$

$$+ a_n^{-4} \iiint K^*\left(\frac{G(x_2)-G(x_0)}{a_n}\right) K_2\left(\frac{G(x_1)-G(x_0)}{a_n}, \frac{F_{x_0}(y_1)-\lambda}{a_n}\right)$$

$$y_1 I(y_2 \leq y_1) dH_n(x_1, y_1) dH_n(x_2, y_2)$$

continued

$$\begin{aligned}
 & + a_n^{-5} \iiint [G_n(x_2) - G_n(x_0) - G(x_2) + G(x_0)] K^{**} \left(\frac{G(x_2) - G(x_0)}{a_n} \right) \\
 & \quad K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 I(y_2 \leq y_1) dH_n(x_1, y_1) \\
 & \quad dH_n(x_2, y_2) \\
 & - a_n^{-3} \iint F_{x_0}(y_1) K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 dH_n(x_1, y_1) \cdot \\
 (4.1.22) \quad \zeta_{n2} &= \frac{a_n^{-6}}{2!} \iiint K^{**} \left(\frac{G(x_2) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) \\
 & \quad y_1 I(y_2 \leq y_1) [G_n(x_2) - G_n(x_0) - G(x_2) + G(x_0)]^2 \\
 & \quad dH_n(x_1, y_1) dH_n(x_2, y_2).
 \end{aligned}$$

$$\begin{aligned}
 (4.1.23) \quad \zeta_{n3} &= \frac{a_n^{-7}}{3!} \iiint K^{***} (\Delta_{x_2, n}) K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) \\
 & \quad y_1 I(y_2 \leq y_1) [G_n(x_2) - G_n(x_0) - G(x_2) + G(x_0)]^3 \\
 & \quad dH_n(x_1, y_1) dH_n(x_2, y_2).
 \end{aligned}$$

$$\begin{aligned}
 (4.1.24) \quad \eta_{nm} &= \frac{a_n^{-2-m}}{m!} \iint [(G_n(x_1) - G_n(x_0) - G(x_1) + G(x_0)) \frac{\partial}{\partial t_1} + \\
 & \quad (F_{n, x_0}(y_1) - F_{x_0}(y_1)) \frac{\partial}{\partial t_2}]^m \\
 & \quad K \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 dH_n(x_1, y_1),
 \end{aligned}$$

for $m = 2, 3$.

$$(4.1.25) \quad \eta_{n4} = \frac{a_n^{-6}}{4!} \iint [(G_n(x_1) - G_n(x_1) - G(x_1) + G(x_0)) \frac{\partial}{\partial t_1} \\ + (F_{n,x_0}(y_1) - F_{x_0}(y_1)) \frac{\partial}{\partial t_2}]^4 K(\Delta_{x_1,n}, \Delta_{y_1,n}) y_1 dH_n(x_1, y_1).$$

LEMMA 4.1.1. Suppose assumptions (4.1.11) to (4.1.13) on the joint distribution H are satisfied. Let $\{a_n\}$ be a sequence of positive numbers such that $a_n \rightarrow 0$ as $n \rightarrow \infty$ and u and v be real numbers such that $|u|, |v| \leq 1$. Then the following Taylor's expansions are valid for sufficiently large n, say, $n > N$:

$$(i) \quad F_{x_0}^{-1}(\lambda + a_n v) = F_{x_0}^{-1}(\lambda) + a_n v D_1 + a_n^2 v^2 D_2 + O(a_n^3),$$

$$(ii) \quad F_{G^{-1}(G(x_0) + a_n u)}^{-1}(F_{x_0}^{-1}(\lambda + a_n v)) = \lambda + a_n u D_3 + a_n v + a_n^2 u^2 D_4 \\ + a_n^2 u v D_5 + O(a_n^3),$$

$$(iii) \quad \frac{h[G^{-1}(G(x_0) + a_n u), F_{x_0}^{-1}(\lambda + a_n v)]}{g[G^{-1}(G(x_0) + a_n u)] f_{x_0}[F_{x_0}^{-1}(\lambda + a_n v)]} \\ = 1 + a_n u D_6 + a_n^2 u^2 D_7 + a_n^2 u v D_8 + O(a_n^3),$$

where $D_1 = D_1(x_0, \lambda) = \frac{1}{f_{x_0}(F_{x_0}^{-1}(\lambda))},$

$$(4.1.26) \quad D_2 = D_2(x_0, \lambda) = -f'_{x_0}(F_{x_0}^{-1}(\lambda)) / 2f_{x_0}^3(F_{x_0}^{-1}(\lambda)),$$

$$D_3 = D_3(x_0, \lambda) = \frac{1}{g^2(x_0)} [H''_1(x_0, F_{x_0}^{-1}(\lambda)) - \lambda g'(x_0)],$$

continued

$$D_4 = D_4(x_0, \lambda) = \frac{1}{2g^4(x_0)} [g(x_0)H_1'''(x_0, F_{x_0}^{-1}(\lambda)) - 3g'(x_0) \\ \times H_1''(x_0, F_{x_0}^{-1}(\lambda)) - \lambda(g''(x_0)g(x_0) - 3(g'(x_0))^2)],$$

$$D_5 = D_5(x_0, \lambda) = \frac{H_{21}''(x_0, F_{x_0}^{-1}(\lambda))}{h(x_0, F_{x_0}^{-1}(\lambda))g(x_0)} - \frac{g'(x_0)}{g^2(x_0)},$$

(4.1.26)

$$D_6 = D_6(x_0, \lambda) = \frac{h_1'(x_0, F_{x_0}^{-1}(\lambda))}{h(x_0, F_{x_0}^{-1}(\lambda))g(x_0)} - \frac{g'(x_0)}{g^2(x_0)},$$

$$D_7 = D_7(x_0, \lambda) = \frac{1}{2g^3(x_0)} \frac{g(x_0)h_1''(x_0, F_{x_0}^{-1}(\lambda)) - 3g'(x_0)h_1'(x_0, F_{x_0}^{-1}(\lambda))}{h(x_0, F_{x_0}^{-1}(\lambda))} \\ - \frac{g''(x_0)g(x_0) - 3(g'(x_0))^2}{g(x_0)},$$

$$D_8 = D_8(x_0, \lambda) = \frac{h_{12}''(x_0, F_{x_0}^{-1}(\lambda)) + g'(x_0)f_{x_0}'(F_{x_0}^{-1}(\lambda))}{h^2(x_0, F_{x_0}^{-1}(\lambda))}.$$

PROOF: (i) Taylor's expansion of $F_{x_0}^{-1}$ about λ yields

$$F_{x_0}^{-1}(\lambda + a_n v) = F_{x_0}^{-1}(\lambda) + \frac{a_n v}{f_{x_0}(F_{x_0}^{-1}(\lambda))} - \frac{a_n^2 v^2}{2f_{x_0}^3(F_{x_0}^{-1}(\lambda))} f'_{x_0}(F_{x_0}^{-1}(\lambda)) + o(a_n^3).$$

(ii) First, note that we can write

$$F_x(y) = \frac{H_1'(x, y)}{g(x)} \quad \text{and hence}$$

$$F_{G^{-1}(G(x_0) + a_n u)}^{-1}(F_{x_0}^{-1}(\lambda + a_n v)) = \frac{H_1'[G^{-1}(G(x_0) + a_n u), F_{x_0}^{-1}(\lambda + a_n v)]}{g[G^{-1}(G(x_0) + a_n u)]}.$$

By the Taylor expansion of the inside functions first,

$$H_1' \left[x_0 + \frac{a_n u}{g(x_0)} - \frac{a_n^2 u^2}{2g^3(x_0)} g'(x_0) + o(a_n^3), F_{x_0}^{-1}(\lambda) + \frac{a_n v}{f_{x_0}(F_{x_0}^{-1}(\lambda))} - \frac{a_n^2 v^2}{2f_{x_0}^3(F_{x_0}^{-1}(\lambda))} f'_{x_0}(F_{x_0}^{-1}(\lambda)) + o(a_n^3) \right]$$

$$g \left[x_0 + \frac{a_n u}{g(x_0)} - \frac{a_n^2 u^2}{2g^3(x_0)} g'(x_0) + o(a_n^3) \right]$$

Now expanding H_1' about $(x_0, F_{x_0}^{-1}(\lambda))$ and g about x_0 ,

$$= \left[H_1'(x_0, F_{x_0}^{-1}(\lambda)) + \left(\frac{a_n u}{g(x_0)} - \frac{a_n^2 u^2}{2g^3(x_0)} g'(x_0) \right) H_1''(x_0, F_{x_0}^{-1}(\lambda)) \right]$$

continued

$$\begin{aligned}
& + \left[\frac{a_n v}{f_{x_0}(F_{x_0}^{-1}(\lambda))} - \frac{a_n^2 v^2}{2f_{x_0}^3(F_{x_0}^{-1}(\lambda))} f'_{x_0}(F_{x_0}^{-1}(\lambda)) \right] H''_{12}(x_0, F_{x_0}^{-1}(\lambda)) + \frac{a_n^2 u^2}{2g^2(x_0)} H'''_1(x_0, F_{x_0}^{-1}(\lambda)) \\
& + \frac{a_n^2 v^2}{2f_{x_0}^2(F_{x_0}^{-1}(\lambda))} H'''_{122}(x_0, F_{x_0}^{-1}(\lambda)) + \frac{a_n^2 uv}{h(x_0, F_{x_0}^{-1}(\lambda))} H'''_{112}(x_0, F_{x_0}^{-1}(\lambda)) + o(a_n^3) \Big] \\
& \left[g(x_0) + \frac{a_n u}{g(x_0)} g'(x_0) - \frac{a_n^2 u^2}{2g^3(x_0)} (g'(x_0))^2 + \frac{a_n^2 u^2}{2g^2(x_0)} g''(x_0) + o(a_n^3) \right]^{-1} \\
& = \left[\frac{H'_1(x_0, F_{x_0}^{-1}(\lambda))}{g(x_0)} + \frac{a_n u}{g^2(x_0)} H''_1(x_0, F_{x_0}^{-1}(\lambda)) + \frac{a_n v}{h(x_0, F_{x_0}^{-1}(\lambda))} H'_{12}(x_0, F_{x_0}^{-1}(\lambda)) \right. \\
& + \frac{a_n^2 u^2}{2g^4(x_0)} [g(x_0) H'''_1(x_0, F_{x_0}^{-1}(\lambda)) - g'(x_0) H''_1(x_0, F_{x_0}^{-1}(\lambda))] + \frac{a_n^2 v^2}{2h(x_0, F_{x_0}^{-1}(\lambda)) f_{x_0}^2(F_{x_0}^{-1}(\lambda))} \\
& \left. [f_{x_0}(F_{x_0}^{-1}(\lambda)) H'''_{122}(x_0, F_{x_0}^{-1}(\lambda)) - f'_{x_0}(F_{x_0}^{-1}(\lambda)) H'_{12}(x_0, F_{x_0}^{-1}(\lambda))] \right. \\
& + \frac{a_n^2 uv H'''_{112}(x_0, F_{x_0}^{-1}(\lambda))}{h(x_0, F_{x_0}^{-1}(\lambda)) g(x_0)} + o(a_n^3) \Big] \left[1 - \frac{a_n u}{g^2(x_0)} g'(x_0) - \frac{a_n^2 u^2}{2g^4(x_0)} [g''(x_0) g(x_0) \right. \\
& \quad \left. - 3(g'(x_0))^2] + o(a_n^3) \right] \\
& = \lambda + \frac{a_n u}{g^2(x_0)} (H''_1(x_0, F_{x_0}^{-1}(\lambda)) - \lambda g'(x_0)) + \frac{a_n v}{h(x_0, F_{x_0}^{-1}(\lambda))} H''_{12}(x_0, F_{x_0}^{-1}(\lambda)) + \frac{a_n^2 u^2}{2g^4(x_0)} \\
& \left[g(x_0) H'''_1(x_0, F_{x_0}^{-1}(\lambda)) - 3H''_1(x_0, F_{x_0}^{-1}(\lambda)) g'(x_0) - \lambda (g''(x_0) g(x_0) - 3(g'(x_0))^2) \right]
\end{aligned}$$

continued

$$\begin{aligned}
& + \frac{a_n^2 v^2}{2h(x_0, F^{-1}(\lambda)) f_{x_0}^2(F^{-1}(\lambda))} \left[f_{x_0}(F^{-1}(\lambda)) H_{122}'''(x_0, F_{x_0}^{-1}(\lambda)) \right. \\
& \left. - f_{x_0}'(F^{-1}(\lambda)) H_{12}''(x_0, F_{x_0}^{-1}(\lambda)) \right] + a_n^2 uv \left[\frac{H_{112}'''(x_0, F_{x_0}^{-1}(\lambda))}{h(x_0, F_{x_0}^{-1}(\lambda)) g(x_0)} - \frac{g'(x_0)}{g^2(x_0)} \right] + O(a_n^3).
\end{aligned}$$

Since

$$H_{12}(x, y) = h(x, y)$$

$$H_{122}'''(x, y) = \frac{\partial}{\partial y} h(x, y) = \frac{\partial}{\partial y} [g(x) f_x(y)] = g(x) f_x'(y),$$

the coefficient of $a_n v$ is equal to one and that of $a_n^2 v^2$ is equal to zero in the above expression and hence is (ii) of the lemma.

(iii) Proceeding on the same lines as in (ii), we have

$$\begin{aligned}
& \frac{h[G^{-1}(G(x_0) + a_n u), F_{x_0}^{-1}(\lambda + a_n v)]}{g[G^{-1}(G(x_0) + a_n u)] f_{x_0}(F_{x_0}^{-1}(\lambda + a_n v))} \\
& = \left[h(x_0, F_{x_0}^{-1}(\lambda)) + \left(\frac{a_n u}{g(x_0)} - \frac{a_n^2 u^2 g'(x_0)}{2g^3(x_0)} \right) h_1'(x_0, F_{x_0}^{-1}(\lambda)) + \left(\frac{a_n v}{f_{x_0}(F_{x_0}^{-1}(\lambda))} \right. \right. \\
& \quad \left. \left. - \frac{a_n^2 v^2}{2f_{x_0}^3(F_{x_0}^{-1}(\lambda))} f_{x_0}'(F_{x_0}^{-1}(\lambda)) \right) h_2'(x_0, F_{x_0}^{-1}(\lambda)) + \frac{a_n^2 u^2}{2g^2(x_0)} h_1''(x_0, F_{x_0}^{-1}(\lambda)) \right]
\end{aligned}$$

continued

$$\begin{aligned}
& + \frac{a_n^{2v^2}}{2f_{x_0}^2(F_{x_0}^{-1}(\lambda))} h_2''(x_0, F_{x_0}^{-1}(\lambda)) + \frac{a_n^{2uv}}{h(x_0, F_{x_0}^{-1}(\lambda))} h_{12}''(x_0, F_{x_0}^{-1}(\lambda)) + o(a_n^3) \Big] \\
& \left[g(x_0) + \frac{a_n^u}{g(x_0)} g'(x_0) - \frac{a_n^{2u^2}}{2g^3(x_0)} (g'(x_0))^2 + \frac{a_n^{2u^2}}{2g^2(x_0)} g''(x_0) + o(a_n^3) \right]^{-1} \left[f_{x_0}(F_{x_0}^{-1}(\lambda)) \right. \\
& + \frac{a_n^v}{f_{x_0}(F_{x_0}^{-1}(\lambda))} f_{x_0}'(F_{x_0}^{-1}(\lambda)) - \frac{a_n^{2v^2}}{2f_{x_0}^3(F_{x_0}^{-1}(\lambda))} (f_{x_0}'(F_{x_0}^{-1}(\lambda)))^2 + \frac{a_n^{2v^2}}{2f_{x_0}^2(F_{x_0}^{-1}(\lambda))} f_{x_0}''(F_{x_0}^{-1}(\lambda)) \\
& \left. + o(a_n^3) \right]^{-1} \\
& = \left[\frac{h(x_0, F_{x_0}^{-1}(\lambda))}{g(x_0) f_{x_0}(F_{x_0}^{-1}(\lambda))} + \frac{a_n^u}{g(x_0)} \frac{h_1'(x_0, F_{x_0}^{-1}(\lambda))}{h(x_0, F_{x_0}^{-1}(\lambda))} + \frac{a_n^v}{f_{x_0}(F_{x_0}^{-1}(\lambda))} \frac{h_2'(x_0, F_{x_0}^{-1}(\lambda))}{h(x_0, F_{x_0}^{-1}(\lambda))} \right. \\
& + \frac{a_n^{2u^2}}{2g^3(x_0)} \left[\frac{g(x_0) h_1''(x_0, F_{x_0}^{-1}(\lambda)) - g'(x_0) h_1'(x_0, F_{x_0}^{-1}(\lambda))}{h(x_0, F_{x_0}^{-1}(\lambda))} \right] + \frac{a_n^{2v^2}}{2f_{x_0}^3(F_{x_0}^{-1}(\lambda))} \\
& \left. \left[\frac{f_{x_0}(F_{x_0}^{-1}(\lambda)) h_2''(x_0, F_{x_0}^{-1}(\lambda)) - f_{x_0}'(F_{x_0}^{-1}(\lambda)) h_2'(x_0, F_{x_0}^{-1}(\lambda))}{h(x_0, F_{x_0}^{-1}(\lambda))} \right] + \frac{a_n^{2uv}}{h^2(x_0, F_{x_0}^{-1}(\lambda))} h_{12}''(x_0, F_{x_0}^{-1}(\lambda)) \right. \\
& \left. + o(a_n^3) \right]
\end{aligned}$$

continued

$$\begin{aligned}
& \left[1 - \frac{a_n^u}{g^2(x_0)} g'(x_0) - \frac{a_n^v}{f_{x_0}^2(F_{x_0}^{-1}(\lambda))} f'_{x_0}(F_{x_0}^{-1}(\lambda)) - \frac{a_n^{2u^2}}{2g^4(x_0)} [g''(x_0)g(x_0) - 3(g'(x_0))^2] \right. \\
& - \frac{a_n^{2v^2}}{2f_{x_0}^4(F_{x_0}^{-1}(\lambda))} [f_{x_0}''(F_{x_0}^{-1}(\lambda))f_{x_0}(F_{x_0}^{-1}(\lambda)) - 3(f'_{x_0}(F_{x_0}^{-1}(\lambda)))^2] \\
& \left. + \frac{a_n^{2uv}}{h^2(x_0, F_{x_0}^{-1}(\lambda))} g'(x_0)f'_{x_0}(F_{x_0}^{-1}(\lambda)) + O(a_n^3) \right] \\
& = 1 + \frac{a_n^u}{g(x_0)} \left[\frac{h'_1(x_0, F_{x_0}^{-1}(\lambda))}{h(x_0, F_{x_0}^{-1}(\lambda))} - \frac{g'(x_0)}{g(x_0)} \right] + \frac{a_n^v}{f_{x_0}(F_{x_0}^{-1}(\lambda))} \left[\frac{h'_2(x_0, F_{x_0}^{-1}(\lambda))}{h(x_0, F_{x_0}^{-1}(\lambda))} - \frac{f'_{x_0}(F_{x_0}^{-1}(\lambda))}{f_{x_0}(F_{x_0}^{-1}(\lambda))} \right] \\
& + \frac{a_n^{2u^2}}{2g^3(x_0)} \left[\frac{g(x_0)h''_1(x_0, F_{x_0}^{-1}(\lambda)) - 3g'(x_0)h'_1(x_0, F_{x_0}^{-1}(\lambda))}{h(x_0, F_{x_0}^{-1}(\lambda))} - \frac{g''(x_0)g(x_0) - 3(g'(x_0))^2}{g(x_0)} \right] \\
& + \frac{a_n^{2v^2}}{2f_{x_0}^3(F_{x_0}^{-1}(\lambda))} \left[\frac{f_{x_0}^{-1}(F_{x_0}^{-1}(\lambda))h''_2(x_0, F_{x_0}^{-1}(\lambda)) - 3f_{x_0}'(F_{x_0}^{-1}(\lambda))h'_2(x_0, F_{x_0}^{-1}(\lambda))}{h(x_0, F_{x_0}^{-1}(\lambda))} \right. \\
& \left. - \frac{f_{x_0}''(F_{x_0}^{-1}(\lambda))f_{x_0}(F_{x_0}^{-1}(\lambda)) - 3(f'_{x_0}(F_{x_0}^{-1}(\lambda)))^2}{f_{x_0}(F_{x_0}^{-1}(\lambda))} \right] + \frac{a_n^{2uv}}{h^2(x_0, F_{x_0}^{-1}(\lambda))} \\
& \quad [h''_{12}(x_0, F_{x_0}^{-1}(\lambda)) + g'(x_0)f'_{x_0}(F_{x_0}^{-1}(\lambda))] + O(a_n^3).
\end{aligned}$$

Since $H(x,y) = g(x) \cdot f_x(y)$,

$$h'_2(x,y) = g(x)f'_x(y) \quad \text{and} \quad h''_2(x,y) = g(x)f''_x(y),$$

we see that the coefficients of $a_n v$ and $a_n^2 v^2$ in the above expression are both equal to zero and consequently (iii) of the lemma follows. $\square$

4.2. THE CONDITIONAL EMPIRICAL PROCESS $\{\pi_{n,x_0}(y), -\infty < y < \infty\}$. In this section we shall apply some results of Stute [18], [19] and the criterion of tightness of a sequence of random functions to show that the supremum of the conditional empirical process defined below ((4.2.1)) is stochastically bounded.

Let $(X_1, Y_1), \dots, (X_n, Y_n)$ be iid with absolutely continuous cdf $H(x,y)$ and joint density $h(x,y)$. Let $G(x), F(y), F_x(y)$ denote the marginal cdf of X , marginal cdf of Y and the conditional cdf of Y given $X = x$ respectively. We define the conditional empirical process

$\pi_{n,x_0}(y)$ by

$$(4.2.1) \quad \pi_{n,x_0}(y) = (na_n)^{1/2} [F_{n,x_0}(y) - F_{x_0}(y)], \quad -\infty < y < \infty,$$

where $F_{n,x_0}(y) = \frac{1}{na_n} \sum_{j=1}^n K^* \left(\frac{G_n(X_j) - G_n(x_0)}{a_n} \right) I(Y_j \leq y)$ as defined by (4.1.2).

LEMMA 4.2.1. Suppose assumptions (4.1.11) to (4.1.15) on the joint distribution H are satisfied and K^* satisfies assumption (4.1.4) with $v^* = 2$. Let $\{a_n\}$ be any sequence of positive numbers such that $a_n \rightarrow 0$, $na_n^4 \rightarrow \infty$ but $na_n^5 \rightarrow 0$. We then have

$$(4.2.2) \quad \sup_{-\infty < y < \infty} |\pi_{n,x_0}(y)| = o_p(1)$$

where $\pi_{n,x_0}(y)$ is defined by (4.2.1).

PROOF: Define $Z_i = F_{x_0}(Y_i)$, $i = 1, \dots, n$ and let $t = F_{x_0}(y)$, $-\infty < y < \infty$.

We observe that

$$0 \leq t \leq 1,$$

$$\text{and } F_{x_0}^*(t) = P(Z \leq t \mid X = x_0) = P(Y \leq F_{x_0}^{-1}(t) \mid X = x_0) = F_{x_0}(F_{x_0}^{-1}(t)) = t.$$

Hence to study the conditional empirical process $\{\pi_{n,x_0}(y): -\infty < y < \infty\}$

obtained from (X_i, Y_i) , $i = 1, 2, \dots, n$ we may as well study the transformed conditional empirical process $\{\pi_{n,x_0}^*(t): 0 \leq t \leq 1\}$ obtained from

(X_i, Z_i) , $i = 1, 2, \dots, n$, namely,

$$(4.2.3) \quad \pi_{n,x_0}^*(t) = (na_n)^{1/2} [F_{n,x_0}^*(t) - F_{x_0}^*(t)] = (na_n)^{1/2} [F_{n,x_0}^*(t) - t],$$

where

$$(4.2.4) \quad F_{n,x_0}^*(t) = \frac{1}{na_n} \sum_{j=1}^n K^*\left(\frac{G_n(X_j) - G_n(x_0)}{a_n}\right) I(Z_j \leq t), \quad 0 \leq t \leq 1.$$

So it suffices to show that $\sup_{0 \leq t \leq 1} |\Pi_{n,x_0}^*(t)| = o_p(1)$. Let H^* and H_n^*

denote, respectively, the joint cdf and the bivariate empirical distribution function of (X, Z) . Observe that (4.2.3) can be rewritten as

$$(4.2.5) \quad \Pi_{n,x_0}^*(t) = (na_n)^{1/2} \left[a_n^{-1} \int \int K^* \left(\frac{G_n(x) - G_n(x_0)}{a_n} \right) I(Z \leq t) dH_n^*(x, z) - t \right].$$

Since K^* is twice differentiable, the Taylor expansion of K^* in (4.2.5) yields

$$\begin{aligned} (4.2.6) \quad \Pi_{n,x_0}^*(t) &= (na_n)^{1/2} \left[a_n^{-1} \int \int K^* \left(\frac{G(x) - G(x_0)}{a_n} \right) I(z \leq t) dH_n^*(x, z) - t \right] \\ &\quad + (na_n)^{1/2} a_n^{-2} \int \int [G_n(x) - G_n(x_0) - G(x) + G(x_0)] K^{*'} \left(\frac{G(x) - G(x_0)}{a_n} \right) \\ &\quad \quad \quad I(z \leq t) dH_n^*(x, z) \\ &\quad + (na_n)^{1/2} a_n^{-3} \int \int [G_n(x) - G_n(x_0) - G(x) + G(x_0)]^2 \frac{K^{*''}(\Delta_{x,n})}{2} \\ &\quad \quad \quad I(z \leq t) dH_n^*(x, z) \\ &= (na_n)^{1/2} a_n^{-1} \int \int K^* \left(\frac{G(x) - G(x_0)}{a_n} \right) I(z \leq t) [dH_n^*(x, z) - dH^*(x, z)] \\ &\quad + (na_n)^{1/2} \left[a_n^{-1} \int \int K^* \left(\frac{G(x) - G(x_0)}{a_n} \right) I(z \leq t) dH^*(x, z) - t \right] \\ &\quad + (na_n)^{1/2} a_n^{-2} \int \int [G_n(x) - G_n(x_0) - G(x) + G(x_0)] K^{*'} \left(\frac{G(x) - G(x_0)}{a_n} \right) \\ &\quad \quad \quad I(z \leq t) dH_n^*(x, z) \end{aligned}$$

$$\begin{aligned}
& + (na_n)^{1/2} a_n^{-3} \iint [G_n(x) - G_n(x_0) - G(x) + G(x_0)]^2 \frac{K^{**}(\Delta_{x,n})}{2} \\
& \quad I(z \leq t) dH_n^*(x, z) \\
& = U_{n,x_0}(t) + V_{n,x_0}(t) + W_{n,x_0}(t) + P_{n,x_0}(t), \quad \text{say,}
\end{aligned}$$

where $\Delta_{x,n}$ lies between $a_n^{-1}(G_n(x) - G_n(x_0))$ and $a_n^{-1}(G(x) - G(x_0))$. We shall

prove the lemma by showing that each one of $\sup_{0 \leq t \leq 1} |U_{n,x_0}(t)|$,

$\sup_{0 \leq t \leq 1} |V_{n,x_0}(t)|$, $\sup_{0 \leq t \leq 1} |W_{n,x_0}(t)|$ and $\sup_{0 \leq t \leq 1} |P_{n,x_0}(t)|$ is $O_p(1)$. We consider

$P_{n,x_0}(t)$ first.

$$(4.2.7) \quad \sup_{0 \leq t \leq 1} |P_{n,x_0}(t)| \leq (na_n)^{1/2} a_n^{-3} \iint [G_n(x) - G_n(x_0) - G(x) + G(x_0)]^2 \frac{K^{**}(\Delta_{x,n})}{2} dH_n^*(x, z)$$

By the same argument as used in lemma 1 of Stute [18], it follows that the R.H.S. of (4.2.7) converges to 0 in probability as $n \rightarrow \infty$. Because of the frequent reliance on this argument with respect to several other terms also in the latter part of this chapter, we reproduce it here for convenience, once for all. Since K^* has compact support, say, $[-1, 1]$ without loss of generality, the above expansion of $F_{n,x_0}^*(t)$ holds true with integration restricted to those x for which $|G_n(x) - G_n(x_0)| < a_n$. The Devoretzky-Keifer-Wolfowitz (1956) bound for the tails of $\sup_x |G_n(x) - G(x)|$

yields that for given $\epsilon > 0$ there exists some finite $C > 0$ such that

$$\sup_x |G_n(x) - G(x)| \leq C \cdot n^{-1/2} \text{ for all } n, \text{ up to an event of probability less}$$

than or equal to ϵ . Let this set be denoted by A_ϵ . On A_ϵ ,

$|G_n(x) - G_n(x_0)| < a_n$ implies

$$|G(x) - G(x_0)| = |G(x) - G_n(x) + G_n(x) - G_n(x_0) + G_n(x_0) - G(x_0)|$$

$$\leq a_n + 2C \cdot n^{-1/2} \leq C_1 a_n$$

for some $C_1 < \infty$ (since $(n^{1/2} a_n)^{-1}$ goes to zero, as $n \rightarrow \infty$).

Lemma 2.3 in Stute [19] asserts that

$$(4.2.8) \quad B_{n,x_0} = \sup_{\{x: |G(x) - G(x_0)| \leq C_1 a_n\}} (n a_n)^{1/2} |G_n(x) - G_n(x_0) - G(x) + G(x_0)|$$

is $O_p(1)$ as $n \rightarrow \infty$. Hence, from (4.2.6) we have

$$(4.2.9) \quad \sup_{0 \leq t \leq 1} |P_{n,x_0}(t)| \leq (n a_n^3)^{-1/2} B_{n,x_0}^2 \iint \frac{K^{**}(\Delta_{x,n})}{2} |dH_n^*(x, z)|, \text{ on the}$$

set A_ϵ . Since K^{**} is bounded, B_{n,x_0} is $O_p(1)$, $n a_n^3 \rightarrow \infty$ and ϵ is arbitrary, the R.H.S. term in (4.2.9) converges to zero in probability as $n \rightarrow \infty$ and hence so does $\sup_{0 \leq t \leq 1} |P_{n,x_0}(t)|$. Thus we have

$$(4.2.10) \quad \sup_{0 \leq t \leq 1} |P_{n,x_0}(t)| = O_p(1).$$

We have
$$\sup_{0 \leq t \leq 1} |W_{n,x_0}(t)| \leq a_n^{-1} \iint \left(\frac{n}{a_n}\right)^{1/2} |G_n(x) - G_n(x_0) - G(x) + G(x_0)|$$

$$|K^{**}\left(\frac{G(x) - G(x_0)}{a_n}\right)| dH_n^*(x, z).$$

By the same argument as used at (4.2.7) above, we can write

$$(4.2.11) \quad \sup_{0 \leq t \leq 1} |W_{n,x_0}(t)| \leq B_{n,x_0} \cdot T_{n,x_0}$$

where B_{n,x_0} is as given by (4.2.8) and

$$T_{n,x_0} = a_n^{-1} \iint |K^{**}\left(\frac{G(x) - G(x_0)}{a_n}\right)| dH_n^*(x, z) = \frac{1}{na_n} \sum_{i=1}^n |K^{**}\left(\frac{G(X_i) - G(x_0)}{a_n}\right)|.$$

Now, by the Markov inequality, for any $M > 0$

$$P(|T_{n,x_0}| > M) \leq \frac{E|T_{n,x_0}|}{M} = \frac{\frac{1}{a_n} \int |K^{**}\left(\frac{G(x) - G(x_0)}{a_n}\right)| dG(x)}{M}$$

$$\leq \frac{\int |K^{**}(u)| du}{M} \leq \frac{C}{M},$$

since K^{**} is bounded and vanishes outside a bounded interval because of assumption (4.1.4). Hence for any $\epsilon > 0$, the R.H.S. quantity can be made $< \epsilon$ by choosing M sufficiently large, which proves that T_{n,x_0} is $O_p(1)$.

Since B_{n,x_0} is also $O_p(1)$, it now follows from (4.2.11) that

$$(4.2.13) \quad \sup_{0 \leq t \leq 1} |W_{n,x_0}(t)| = O_p(1).$$

Observe that
$$a_n^{-1} \int K^{**}\left(\frac{G(x) - G(x_0)}{a_n}\right) dG(x) = \frac{(1 - G(x_0))/a_n}{-G(x_0)/a_n} \int K^{**}(u) du.$$

Since K^* has bounded support $[-1, 1]$ $0 < G(x_0) < 1$ and $a_n \rightarrow 0$ as

$n \rightarrow \infty$, the last integral is equal to one for sufficiently large n , say, for $n > n^*$. Hence we may write for $n > n^*$

$$(4.2.14) \quad v_{n,x_0}(t) = (na_n)^{1/2} a_n^{-1} \int K^*\left(\frac{G(x)-G(x_0)}{a_n}\right) [F_x^*(t)-t] dG(x) \\ = (na_n)^{1/2} \int_{-1}^1 K^*(u) \left[F_{G^{-1}(G(x_0)+a_n u)}^*(t) - t \right] du$$

where the last equality follows upon transformation of variables. Now, since H^* is four times differentiable, which follows from assumption (4.1.15), Taylor's expansion of $F_{G^{-1}(G(x_0)+a_n u)}^*(t)$ as in (iii) of lemma 4.1.1 yields

$$(4.2.15) \quad v_{n,x_0}(t) = (na_n)^{1/2} \int_{-1}^1 K^*(u) [F_{x_0}^*(t) + a_n u D_3(x_0, t) + a_n^2 u^2 D_4(x_0, t) + O(a_n^3) - t] du$$

where (see (4.1.26))

$$D_3(x_0, t) = \frac{1}{g^2(x_0)} [H_1^{*''}(x_0, t) - t g'(x_0)]$$

and

$$D_4(x_0, t) = \frac{1}{2g^4(x_0)} [g(x_0) H_1^{*'''}(x_0, t) - g'(x_0) H_1^{*''}(x_0, t) - t g''(x_0) g(x_0) \\ - 3t (g'(x_0))^2 - 2g'(x_0) H_1^{*''}(x_0, t)].$$

Since $\sup_{0 \leq t \leq 1} |D_4(x_0, t)| < \infty$ (by assumptions (4.1.11) to (4.1.15)), $F_{x_0}^*(t) = t$, $\int u K^*(u) du = 0$ and $na_n^5 \rightarrow 0$ as $n \rightarrow \infty$, it follows from (4.2.15) that

$$\sup_{0 \leq t \leq 1} |v_{n,x_0}(t)| = o(1).$$

Hence it remains to show that $\sup_{0 \leq t \leq 1} |u_{n,x_0}(t)| = o_p(1)$. This we shall do by

proving the tightness of the sequence $\{U_{n,x_0}\}$ on the $D[0,1]$. The required result will then follow from Theorem 15.2 of [2].

The tightness will be proved by verifying the conditions of Theorem 15.5 of [2]. Since $U_{n,x_0}(0) = 0$ for all n , by the continuity assumption on F , condition (i) of Theorem 15.5 of [2] is trivially satisfied. Hence, it suffices to show that condition (ii) of Theorem 15.5 of [2] is also satisfied, namely, for each positive ϵ and η there exists a δ , $0 < \delta < 1$, such that

$$(4.2.16) \quad P\{W(U_{n,x_0}, \delta) > \epsilon\} \leq \eta$$

for all sufficiently large n , where $W_U(\delta) = W(U, \delta)$ is the modulus of continuity of the function U as defined by (8.1) of [2].

Now fix ϵ and η . Suppose $0 \leq s \leq t \leq 1$. From (4.2.6) we have

$$(4.2.17) \quad U_{n,x_0}(t) - U_{n,x_0}(s) = (na_n)^{-1/2} \sum_{i=1}^n \left[K^* \left(\frac{G(X_i) - G(x_0)}{a_n} \right) I(s \leq Z_i \leq t) \right. \\ \left. - \int K^* \left(\frac{G(x) - G(x_0)}{a_n} \right) I(s \leq z \leq t) dH^*(x, z) \right]$$

$$= (na_n)^{-1/2} \sum_{i=1}^n (T_i - ET_i)$$

where

$$T_i = K^* \left(\frac{G(X_i) - G(x_0)}{a_n} \right) I(s \leq Z_i \leq t)$$

Since T_i 's are all iid rv's

$$(4.2.18) \quad E(U_{n,x_0}(t) - U_{n,x_0}(s))^4 = \frac{1}{n^2 a_n^2} [n E(T_i - ET_i)^4 + 3n(n-1) \{E(T_i - ET_i)^2\}^2].$$

Since T_1 's are all nonnegative rv's, we see that

$$\begin{aligned}
 (4.2.19) \quad E(T_1 - ET_1)^4 &\leq E(T_1^4) + 6E(T_1^2) E^2(T_1) \\
 &= \int K^{*4} \left(\frac{G(x_1) - G(x_0)}{a_n} \right) (F_x^*(t) - F_x^*(s)) dG(x) \\
 &\quad + 6 \left(\int K^{*2} \left(\frac{G(x) - G(x_0)}{a_n} \right) (F_x^*(t) - F_x^*(s)) dG(x) \right) \\
 &\quad \left(\int K^* \left(\frac{G(x) - G(x_0)}{a_n} \right) (F_x^*(t) - F_x^*(s)) dG(x) \right)^2.
 \end{aligned}$$

Since the conditional density $f_x(y)$ is assumed to be bounded uniformly in x and y , we may write

$$(4.2.20) \quad |F_x^*(t) - F_x^*(s)| \leq C_1(t-s) \text{ where } C_1 = \sup_{x,z} |f_x^*(z)|.$$

By (4.2.20) and the fact that K^* is bounded it follows from (4.2.19) that

$$\begin{aligned}
 E(T_1 - ET_1)^4 &\leq C_1(t-s) \int \left[K^* \left(\frac{G(x) - G(x_0)}{a_n} \right) \right]^4 dG(x) + C_2(t-s) \int \left[K^* \left(\frac{G(x) - G(x_0)}{a_n} \right) \right]^2 \\
 &\quad dG(x) \\
 &\leq C_3 a_n (t-s)
 \end{aligned}$$

where the last inequality follows upon transformation of variables. Note that the constant C_3 depends on the bounds of K^* and $f_x^*(z)$ only. We also have,

$$\begin{aligned}
 (4.2.22) \quad E(T_1 - ET_1)^2 &\leq E(T_1^2) = \int \left[K^* \left(\frac{G(x) - G(x_0)}{a_n} \right) \right]^2 (F_x^*(t) - F_x^*(s)) dG(x) \\
 &\leq C_4 a_n (t-s),
 \end{aligned}$$

where the constant $C_4 > 0$ again depends on the bounds of K^* and $f_x^*(z)$ only.

Now, from (4.2.19), (4.2.21) and (4.2.22), we have

$$\begin{aligned} E(U_{n,x_0}(t) - U_{n,x_0}(s))^4 &\leq \frac{C_3}{na_n} (t-s) + 3C_4^2 (t-s)^2 \\ &\leq \frac{C_5}{na_n} (t-s) + C_5 (t-s)^2. \end{aligned}$$

Hence for any $0 \leq s, t \leq 1$,

$$(4.2.23) \quad E|U_{n,x_0}(t) - U_{n,x_0}(s)|^4 \leq \frac{C_5}{na_n} |t-s| + C_5 |t-s|^2,$$

and, if $\frac{\epsilon}{na_n} < |t-s|$ (with $\epsilon < 1$), we have

$$(4.2.24) \quad E|U_{n,x_0}(t) - U_{n,x_0}(s)|^4 \leq \frac{2C_5}{\epsilon} (t-s)^2,$$

where the constant C_5 depends on the bounds of K^* and $f_x^*(z)$ only. Now suppose that p is a number satisfying $\frac{\epsilon}{na_n} \leq p$, and consider the rv's

$$U_{n,x_0}(s+ip) - U_{n,x_0}(s+(i-1)p), \quad i = 1, 2, \dots, m, \quad \text{where } m \text{ is a positive integer.}$$

By theorem 12.2 of [2]

$$(4.2.25) \quad P\left\{\max_{1 \leq i \leq m} |U_{n,x_0}(s+ip) - U_{n,x_0}(s+(i-1)p)| \geq \lambda\right\} \leq \frac{C^*}{\epsilon \lambda^4} m^2 p^2.$$

For $s \leq t \leq s+p$,

$$\begin{aligned}
 (4.2.26) \quad U_{n,x_0}(t) - U_{n,x_0}(s) &= (na_n)^{-1/2} \sum_{i=1}^n \left[K^* \left(\frac{G(x_i) - G(x_0)}{a_n} \right) I(Z_i \leq t) \right. \\
 &\quad \left. - \int K^* \left(\frac{G(x) - G(x_0)}{a_n} \right) I(Z \leq t) dH^*(x, z) \right] \\
 &\quad - U_{n,x_0}(s) \\
 &\leq (na_n)^{-1/2} \sum_{i=1}^n \left[K^* \left(\frac{G(x_i) - G(x_0)}{a_n} \right) I(Z_i \leq s+p) \right. \\
 &\quad \left. - \int K^* \left(\frac{G(x) - G(x_0)}{a_n} \right) I(Z \leq s+p) dH^*(x, z) \right. \\
 &\quad + \int K^* \left(\frac{G(x) - G(x_0)}{a_n} \right) I(Z \leq s+p) dH^*(x, z) \\
 &\quad \left. - \int K^* \left(\frac{G(x) - G(x_0)}{a_n} \right) I(Z \leq t) dH^*(x, z) \right] - U_{n,x_0}(s) \\
 &= [U_{n,x_0}(s+p) - U_{n,x_0}(s)] + \left(\frac{n}{a_n} \right)^{1/2} \int K^* \left(\frac{G(x) - G(x_0)}{a_n} \right) I(t \leq Z \leq s+p) dH^*(x, z) \\
 &\leq |U_{n,x_0}(s+p) - U_{n,x_0}(s)| + \left(\frac{n}{a_n} \right)^{1/2} \int K^* \left(\frac{G(x) - G(x_0)}{a_n} \right) I(s \leq Z \leq s+p) dH^*(x, z) .
 \end{aligned}$$

Also,

$$\begin{aligned}
 (4.2.27) \quad & - [U_{n,x_0}(t) - U_{n,x_0}(s)] \leq \left(\frac{n}{a_n}\right)^{1/2} \int K^* \left(\frac{G(x) - G(x_0)}{a_n} \right) I(s \leq z \leq t) dH^*(x, z) \\
 & - (na_n)^{-1/2} \left[\sum_{i=1}^n K^* \left(\frac{G(X_1) - G(x_0)}{a_n} \right) I(Z_1 \leq t) \right. \\
 & \quad \left. - \sum_{i=1}^n K^* \left(\frac{G(X_1) - G(x_0)}{a_n} \right) I(Z_1 \leq s) \right] \\
 & = \left(\frac{n}{a_n}\right)^{1/2} \int K^* \left(\frac{G(x) - G(x_0)}{a_n} \right) I(s \leq z \leq t) dH^*(x, z) \\
 & \quad - (na_n)^{-1/2} \sum_{i=1}^n K^* \left(\frac{G(X_1) - G(x_0)}{a_n} \right) I(s \leq Z_1 \leq t) \\
 & \leq \left(\frac{n}{a_n}\right)^{1/2} \int K^* \left(\frac{G(x) - G(x_0)}{a_n} \right) I(s \leq z \leq s+p) dH^*(x, z) \\
 & \leq \left(\frac{n}{a_n}\right)^{1/2} \int K^* \left(\frac{G(x) - G(x_0)}{a_n} \right) I(s \leq z \leq s+p) dH^*(x, z) \\
 & \quad + |U_{n,x_0}(s+p) - U_{n,x_0}(s)|.
 \end{aligned}$$

Hence from (4.2.26) and (4.2.27) we have

$$(4.2.28) \quad |U_{n,x_0}(t) - U_{n,x_0}(s)| \leq |U_{n,x_0}(s+p) - U_{n,x_0}(s)| + \left(\frac{n}{a_n}\right)^{1/2} \int K^* \left(\frac{G(x) - G(x_0)}{a_n} \right) I(s \leq z \leq s+p) dH^*(x, z)$$

for $s \leq t \leq s+p$.

But the second term on the R.H.S. of (4.2.28) is

$$= \left(\frac{n}{a_n}\right)^{1/2} \int K^* \left(\frac{G(x) - G(x_0)}{a_n} \right) (F_x^*(s+p) - F_x^*(s)) dG(x) \\ \leq (na_n)^{1/2} p C_1,$$

by (4.2.20) and the fact that K^* is a density function. Hence we have

$$(4.2.29) \quad |U_{n,x_0}(t) - U_{n,x_0}(s)| \leq |U_{n,x_0}(s+p) - U_{n,x_0}(s)| + (na_n)^{1/2} p C_1$$

for $s \leq t \leq s+p$. But (4.2.29) implies

$$(4.2.30) \quad \sup_{s \leq t \leq s+mp} |U_{n,x_0}(t) - U_{n,x_0}(s)| \leq 3 \max_{i \leq m} |U_{n,x_0}(s+ip) - U_{n,x_0}(s)| (na_n)^{1/2} p C_1.$$

Now suppose that $\frac{\epsilon}{na_n} < p < \frac{\epsilon}{(na_n)^{1/2} C_1}$.

Then (4.2.25) applies and it follows by (4.2.30) that

$$(4.2.31) \quad P \left\{ \sup_{s \leq t \leq s+mp} |U_{n,x_0}(t) - U_{n,x_0}(s)| \geq 4\epsilon \right\} \leq \frac{C^*}{\epsilon^5} m^2 p^2.$$

Choose δ so that $\frac{C^* \delta}{\epsilon^5} < n$. Then from (4.2.31) we have

$$(4.2.32) \quad P \left\{ \sup_{s \leq t \leq s+\delta} |U_{n,x_0}(t) - U_{n,x_0}(s)| \geq 4\epsilon \right\} \leq n \delta,$$

provided there exists a 'p' and an integer 'm' such that $mp = \delta$, and

$\frac{\varepsilon}{na_n} < p < \frac{\varepsilon}{(na_n)^{1/2}C_1}$ is satisfied or equivalently, there exists an integer 'm'

with $(\delta/\varepsilon)(na_n)^{1/2}C_1 < m < (\delta/\varepsilon)na_n$, which is true for all sufficiently large n , since $na_n \rightarrow \infty$. For given ε and n , therefore, there exists a δ such that (4.2.32) holds for sufficiently large n . But this implies (4.2.16) by the Corollary to Theorem 8.3 of [2]. This completes the proof of tightness of $\{U_{n,x_0}\}$ and hence the lemma. $\square$

4.3 ASYMPTOTIC EQUIVALENCE OF $Q_{n,x_0}(\lambda)$ AND $S_{n,x_0}(\lambda)$ IN DISTRIBUTION. In

this section we shall prove that $Q_{n,x_0}(\lambda)$ and $S_{n,x_0}(\lambda)$, as given by (4.1.20) and (4.1.21) respectively, are asymptotically equivalent in distribution by showing that $Q_{n,x_0}(\lambda) - S_{n,x_0}(\lambda)$ converges to zero in probability as $n \rightarrow \infty$.

From (4.1.20) we have

$$Q_{n,x_0}(\lambda) - S_{n,x_0}(\lambda) = \sum_{m=2}^3 \tau_{nm} + \sum_{m=2}^4 \eta_{nm}$$

where τ_{nm} 's are given by (4.1.22) to (4.1.23) and η_{nm} are given by (4.1.24) to (4.1.25). Hence it suffices to show that each of the terms

τ_{n2} , τ_{n3} , η_{n2} , η_{n3} and η_{n4} converge to zero in probability as $n \rightarrow \infty$.

Lemma 4.2.1 is particularly useful in this regard.

LEMMA 4.3.1. (a) Suppose assumptions (4.1.11) to (4.1.13) are satisfied and let $\{a_n\}$ be a sequence of positive numbers with $a_n \rightarrow 0$ and $na_n^4 \rightarrow \infty$. We then have

$$\tau_{nm} \xrightarrow{p} 0 \text{ as } n \rightarrow \infty \text{ for } m = 2, 3$$

where τ_{n2} and τ_{n3} are given by (4.1.22) and (4.1.23) respectively.

(b) Suppose assumptions (4.1.11) to (4.1.15) are satisfied and let $\{a_n\}$ be a sequence of positive numbers with $a_n \rightarrow 0$, $na_n^4 \rightarrow \infty$ but $na_n^5 \rightarrow 0$. We then have

$$\eta_{nm} \xrightarrow{pr} 0 \text{ as } n \rightarrow \infty \text{ for } m = 2, 3 \text{ and } 4,$$

where η_{nm} , $m = 2, 3$ are given by (4.1.24) and η_{n4} is given by (4.1.25).

PROOF: (a) Since K^{**} is bounded, from (4.1.22) we have

$$|\zeta_{n2}| \leq Ca_n^{-6} \iiint [G_n(x_2) - G_n(x_0) - G(x_2) + G(x_0)]^2 \\ |K_2'(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n})| |y_1| dH_n(x_1, y_1) dH_n(x_2, y_2).$$

By the same argument as used at (4.2.7), we further have

$$(4.3.1) \quad |\zeta_{n2}| \leq \frac{C}{na_n^5} B_{n,x_0}^2 \iint |K_2'(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n}) y_1| dH_n(x_1, y_1)$$

where B_{n,x_0} is as given by (4.2.8)

Now, we can write

$$(4.3.2) \quad \frac{1}{na_n^5} \iint |K_2'(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n}) y_1| dH_n(x_1, y_1) \\ = \frac{1}{n} \sum_{i=1}^n |\frac{1}{na_n^5} K_2'(\frac{G(X_i) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n}) y_1| \\ = \frac{1}{n} \sum T_i = \bar{T}_n$$

and by the Markov inequality, we have

$$(4.3.3) \quad p(\bar{T}_n > \epsilon) \leq \frac{E(T_1)}{\epsilon}.$$

$$\begin{aligned} \text{But } E(T_1) &= \frac{1}{na_n^5} \iint |K_2'(\frac{G(x_1)-G(x_0)}{a_n}, \frac{F_{x_0}(y_1)-\lambda}{a_n})| |y_1| h(x_1, y_1) dx_1 dy_1 \\ &\leq \frac{1}{na_n^3} \iint |K_2'(u_1, v_1)| |F_{x_0}^{-1}(\lambda + a_n v_1)| \frac{h[G^{-1}(G(x_0) + a_n u_1), F_{x_0}^{-1}(\lambda + a_n v_1)]}{g[G^{-1}(G(x_0) + a_n u_1)] f_{x_0}[F_{x_0}^{-1}(\lambda + a_n v_1)]} \\ &\quad \cdot du_1 dv_1 \end{aligned}$$

where the last inequality follows by transformation of variables $\frac{G(x_1)-G(x_0)}{a_n} = u_1$

and $\frac{F_{x_0}(y_1)-\lambda}{a_n} = v_1$, and using the fact that $P\{G^{-1}(G(X)) \neq X, F_{x_0}^{-1}(F_{x_0}(Y)) \neq Y\} = 0$. Hence

using the Taylor expansions from lemma 4.1.1, we see that

$$\begin{aligned} (4.3.4) \quad E(T_1) &\leq \frac{1}{na_n^3} \iint |K_2'(u_1, v_1)| [F_{x_0}^{-1}(\lambda) + a_n v_1 D_1 + a_n^2 v_1^2 D_2 + O(a_n^3)] \\ &\quad [1 + a_n u_1 D_6 + a_n^2 u_1^2 D_7 + a_n^2 u_1 v_1 D_8 + O(a_n^3)] |du_1 dv_1 \\ &= O(\frac{1}{na_n^3}). \end{aligned}$$

Since $na_n^3 \rightarrow \infty$, from (4.3.3) and (4.3.4) we have

$$(4.3.5) \quad \bar{T}_n \xrightarrow{pr} 0, \text{ as } n \rightarrow \infty.$$

Since B_{n, x_0} is $O_p(1)$ up to an event of probability less than ϵ and ϵ is arbitrary, from (4.3.1) and (4.3.5) it now follows that

$$\zeta_{n2} \xrightarrow{pr} 0 \text{ as } n \rightarrow \infty.$$

The argument used at (4.2.7), when applied to (4.1.23) yields

$$|\zeta_{n3}| \leq \frac{1}{n^{3/2} a_n^{11/2}} B_{n,x_0}^3 \iiint |K^{***}(\Delta_{x_2,n}) K_2'(\frac{F(x_1)-G(x_0)}{a_n}, \frac{F_{x_0}(y_1)-\lambda}{a_n})| \\ |y_1| I(y_2 \leq y_1) dH_n(x_1, y_1) dH_n(x_2, y_2).$$

Since K^{***} and K_2' are bounded we can further write

$$(4.3.5) \quad |\zeta_{n3}| \leq \frac{C}{n^{3/2} a_n^{11/2}} B_{n,x_0}^3 \iint |y_1| dH_n(x_1, y_1).$$

Since $na_n^4 \rightarrow \infty$, $\lim_{n \rightarrow \infty} \sup \iint |y_1| dH_n(x_1, y_1) < \infty$ with probability one, and B_{n,x_0} is $O_p(1)$ up to an event of probability less than ϵ , but ϵ is arbitrary, it follows that

$$\zeta_{n3} \xrightarrow{pr} 0 \text{ as } n \rightarrow \infty.$$

(b) From (4.1.24), observe that the terms involved in η_{nm} , for a fixed $m(m=2,3)$, are of the following three different types:

$$\eta_{nm}(A) = \frac{a_n^{-2-m}}{m!} \iint [G_n(x) - G_n(x_0) - G(x) + G(x_0)]^m K_1^{(m)}(\frac{G(x)-G(x_0)}{a_n}, \frac{F_{x_0}(y)-\lambda}{a_n}) \\ y dH_n(x, y),$$

$$\eta_{nm}(B) = \frac{a_n^{-2-m}}{m!} \iint [F_{n,x_0}(y) - F_{x_0}(y)]^m K_2^{(m)}(\frac{G(x)-G(x_0)}{a_n}, \frac{F_{x_0}(y)-\lambda}{a_n}) y dH_n(x, y),$$

$$\text{and } \eta_{nm}(C) = \frac{a_n^{-2-m}}{m!} \iint [G_n(x) - G_n(x_0) - G(x) + G(x_0)]^{m_1} [F_{n,x_0}(y) - F_{x_0}(y)]^{m_2} \\ K_{12}^{(m_1, m_2)}(\frac{G(x)-G(x_0)}{a_n}, \frac{F_{x_0}(y)-\lambda}{a_n}) y dH_n(x, y)$$

where $m_1 + m_2 = m$, $1 < m_1, m_2 < m$,

$k_{1-}^{(m)}(\theta_1, \theta_2)$ is the m^{th} order partial derivative of $K(\dots)$ with respect to the i^{th} argument evaluated at (θ_1, θ_2) , $i=1,2$,

$k_{12}^{(m_1, m_2)}(\theta_1, \theta_2)$ is the m^{th} mixed derivative of $K(\dots)$, obtained by differentiating m_1 times with respect to the first argument and m_2 times with respect to the second argument, and evaluated at (θ_1, θ_2) .

Using the argument applied at (4.2.7), we have

$$|\eta_{nm}(A)| \leq \left(\frac{1}{na_n^{1+\frac{4}{m}}} \right)^{\frac{m}{2}} B_{n, x_0}^m \iint |K_1^{(m)} \left(\frac{G(x) - G(x_0)}{a_n}, \frac{F_{x_0}^{(y)} - \lambda}{a_n} \right) y| dH_n(x, y).$$

Since $na_n^{1+\frac{4}{m}} \rightarrow \infty$ for $m = 2, 3$, by the same argument as used in part (a), for the term ζ_{n3} , we see that

$$\eta_{nm}(A) \rightarrow 0 \text{ is probability, for } m = 2, 3.$$

$$\text{Now, } |\eta_{nm}(B)| \leq \frac{1}{n^{m/2} a_n^{\frac{3m+4}{2}}} L_{n, x_0}^m \iint |K_2^{(m)} \left(\frac{G(x) - G(x_0)}{a_n}, \frac{F_{x_0}^{(y)} - \lambda}{a_n} \right) y| dH_n(x, y),$$

where

$$(4.3.6) \quad L_{n, x_0} = \sup_{-\infty < y < \infty} (na_n)^{1/2} |F_{n, x_0}(y) - F_{x_0}(y)|$$

From lemma 4.2.1, we have that $L_{n,x_0} = O_p(1)$. Applying the same argument

as used in part (a) for the term ζ_{n2} and observing that $n^{\frac{m}{2}} a_n^{\frac{3m}{2}} \rightarrow \infty$ for $m = 2, 3$, it follows that

$$\frac{\eta_{nm}(c)}{nm} \rightarrow 0 \text{ in probability, for } m = 2, 3.$$

Using the argument applied at (4.2.7) and lemma 4.2.1, we have

$$(4.3.7) \quad |\eta_{nm}(c)| \leq \frac{a_n^{-2-m}}{m!} \left(\frac{a_n}{n}\right)^{\frac{m_1}{2}} \left(\frac{1}{na_n}\right)^{\frac{m_2}{2}} B_{n,x_0}^{m_1} L_{n,x_0}^{m_2} \iint |K_2^{(m_1, m_2)} \left(\frac{G(x)-G(x_0)}{a_n}\right) y| dH_n(x, y),$$

where both B_{n,x_0} and L_{n,x_0} are $O_p(1)$. Consider the case $m_1 = 1$, $m_2 = m-1$ in (4.3.7) first. Then (4.3.7) becomes

$$(4.3.8) \quad |\eta_{nm}(c)| \leq \frac{1}{n^{\frac{m}{2}} a_n^{\frac{3m+2}{2}}} B_{n,x_0} L_{n,x_0}^{m-1} \iint |K_2^{(1, m-1)} \left(\frac{G(x)-G(x_0)}{a_n}, \frac{F_{x_0}(y)-\lambda}{a_n}\right) y| dH_n(x, y).$$

Again, by the same argument as used in part (a) for the term ζ_{n3} and

observing that $n^{\frac{m}{2}} a_n^{\frac{3m+2}{2}} \rightarrow \infty$ for $m = 2, 3$, it follows that

$$\eta_{nm}(c) \rightarrow 0 \text{ in probability, for } m = 2, 3 \text{ with } m_1 = 1, m_2 = m-1.$$

For all the other values of m_1 and m_2 such that $m_1 + m_2 = m$, $m_1 > 1$, (for each fixed m), the constant term on the R.H.S. of (4.3.8) will be of the form $\frac{1}{n^{(m-1)/2} a_n^\alpha}$ with $\alpha < \frac{3m}{2}$. Hence, convergence of $n^{1/2} a_n \cdot \eta_{nm}(C)$ to zero is probability, with those values of m_1, m_2 automatically follows, once, it is established with $m_1 = 1, m_2 = m-1$. Therefore, we have

$$\eta_{nm}(C) \rightarrow 0 \text{ in probability, for } m = 2, 3.$$

It remains to show that η_{n4} goes to zero in probability. First, from (4.1.25) observe that η_{n4} again has the same types of terms as η_{n2} or

$$\eta_{n3}, \text{ with } K_1^{(m)} \left(\frac{G(x) - G(x_0)}{a_n}, \frac{F_{x_0}(y) - \lambda}{a_n} \right), K_2^{(m)} \left(\frac{G(x) - G(x_0)}{a_n}, \frac{F_{x_0}(y) - \lambda}{a_n} \right) \text{ and}$$

$$K_{12}^{(m_1, m_2)} \left(\frac{G(x) - G(x_0)}{a_n}, \frac{F_{x_0}(y) - \lambda}{a_n} \right) \text{ replaced by } K_1^{(m)}(\Delta_{x,n}, \Delta_{y,n}), K_2^{(m)}(\Delta_{x,n}, \Delta_{y,n})$$

and $K_{12}^{(m_1, m_2)}(\Delta_{x,n}, \Delta_{y,n})$ respectively, namely,

$$\eta_{n4}(A) = \frac{a_n^{-6}}{4!} \iint [G_n(x) - G_n(x_0) - G(x) + G(x_0)]^4 K_1^{(4)}(\Delta_{x,n}, \Delta_{y,n}) y dH_n(x, y),$$

$$\eta_{n4}(B) = \frac{a_n^{-6}}{4!} \iint [F_{n, x_0}(y) - F_{x_0}(y)]^4 K_2^{(4)}(\Delta_{x,n}, \Delta_{y,n}) y dH_n(x, y),$$

$$\text{and } \eta_{n4}(\bar{C}) = \frac{a_n^{-6}}{4!} \iint [G_n(x) - G_n(x_0) - G(x) + G(x_0)]^{m_1} [F_{n,x_0}(y) - F_{x_0}(y)]^{m_2}$$

$$K_{12}^{(m_1, m_2)}(\Delta_{x,n}, \Delta_{y,n}) y dH_n(x, y)$$

where $m_1 + m_2 = 4$, $m_1 \geq 1$ and $m_2 \geq 1$.

The convergence in probability to zero of $\eta_{n4}(A)$ follows on the same lines as that of ζ_{n3} in part (a) above. Now

$$(4.3.9) \quad |\eta_{n4}(B)| \leq \frac{1}{n^2 a_n^8} L_{n,x_0}^4 \iint |K_2^{(4)}(\Delta_{x,n}, \Delta_{y,n}) y| dH_n(x, y)$$

where L_{n,x_0} is as defined by (4.2.29), and is $O_p(1)$ by lemma 4.2.1.

Since $K_2^{(4)}$ is bounded, $\lim_{n \rightarrow \infty} \sup \iint |y| dH_n(x, y) < \infty$ and $na_n^4 \rightarrow \infty$, from (4.3.9)

it follows that $\eta_{n4}(B) \rightarrow 0$ in probability.

As before (in the case of η_{nm} for $m < 4$), for $\eta_{n4}(C)$ we need

only consider the case $m_1 = 1$ and $m_2 = m-1 = 3$. The same argument leading to (4.3.8) yields

$$(4.3.10) \quad |\eta_{n4}(C)| \leq \frac{1}{n^2 a_n^7} B_{n,x_0} L_{n,x_0}^3 \iint |K_2^{(4)}(\Delta_{x,n}, \Delta_{y,n}) y| dH_n(x, y).$$

It now follows from (4.3.10) that $\eta_{n4}(C)$ converges to zero in probability

just as that of $\eta_{n4}(B)$ followed from (4.3.8). This completes the proof of

part (b) of the lemma. Hence the lemma. $\square$

4.4 ASYMPTOTIC VARIANCE OF $n^{1/2} a_n \cdot S_{n,x_0}^{(\lambda)}$ — In this section we shall show

that $\text{var}(n^{1/2} a_n \cdot S_{n,x_0}^{(\lambda)})$ converges to a finite value as $n \rightarrow \infty$, where

$S_{n,x_0}^{(\lambda)}$ is as given by (4.1.21). First observe that $S_{n,x_0}^{(\lambda)}$ may be

rewritten as

$$\begin{aligned}
 (4.4.1) \quad S_{n,x_0}^{(\lambda)} &= \frac{1}{na_n^2} \sum_{i=1}^n K\left(\frac{G(X_i) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_i) - \lambda}{a_n}\right) Y_i \\
 &\quad + \frac{1}{na_n^3} \sum_{i=1}^n [G_n(X_i) - G_n(x_0) - G(X_i) + G(x_0)] \\
 &\quad \quad \quad \cdot K_1\left(\frac{G(X_i) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_i) - \lambda}{a_n}\right) Y_i \\
 &\quad + \frac{1}{n^2 a_n^4} \sum_{i=1}^n \sum_{j=1}^n K^*\left(\frac{G(X_j) - G(x_0)}{a_n}\right) K_2\left(\frac{G(X_i) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_i) - \lambda}{a_n}\right) \\
 &\quad \quad \quad Y_i I(Y_j < Y_i) \\
 &\quad + \frac{1}{n^2 a_n^5} \sum_{i=1}^n \sum_{j=1}^n [G_n(X_j) - G_n(x_0) - G(X_i) + G(x_0)] \\
 &\quad \quad \quad \cdot K^*\left(\frac{G(X_j) - G(x_0)}{a_n}\right) K_2\left(\frac{G(X_i) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_i) - \lambda}{a_n}\right) \\
 &\quad \quad \quad Y_i I(Y_j < Y_i) \\
 &\quad - \frac{1}{na_n^3} \sum_{i=1}^n F_{x_0}(Y_i) K_2\left(\frac{G(X_i) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_i) - \lambda}{a_n}\right) Y_i \\
 &= I_1 + I_2 + I_{31} + I_{32} - I_{30}.
 \end{aligned}$$

We shall further rewrite $I_{31} = I_{311} + I_{312}$ and $I_{32} = I_{321} + I_{322}$ so that

$$(4.4.2) \quad S_{n,x_0}(\lambda) = I_1 + I_2 + I_{311} + I_{312} + I_{321} + I_{322} - I_{30}$$

where I_{311} , I_{312} , I_{321} and I_{322} are given below by (4.4.3), (4.4.4), (4.4.5) and (4.4.6) respectively:

$$(4.4.3) \quad I_{311} = \frac{1}{n^2 a_n^4} \sum_{i=1}^n K^* \left(\frac{G(X_i) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(X_i) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_i) - \lambda}{a_n} \right) Y_i.$$

$$(4.4.4) \quad I_{312} = \frac{1}{n^2 a_n^4} \sum_{i \neq j} K^* \left(\frac{G(X_j) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(X_i) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_i) - \lambda}{a_n} \right) Y_i I(Y_j < Y_i).$$

$$(4.4.5) \quad I_{321} = \frac{1}{n^2 a_n^5} \sum_{i=1}^n [G_n(X_i) - G_n(x_0) - G(X_i) + G(x_0)] K^* \left(\frac{G(X_i) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(X_i) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_i) - \lambda}{a_n} \right) Y_i.$$

$$(4.4.6) \quad I_{322} = \frac{1}{n^2 a_n^5} \sum_{i \neq j} [G_n(X_j) - G_n(x_0) - G(X_j) + G(x_0)] K^* \left(\frac{G(X_j) - G(x_0)}{a_n} \right)$$

$$K_2 \left(\frac{G(X_i) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_i) - \lambda}{a_n} \right) Y_i I(Y_j < Y_i).$$

LEMMA 4.4.1 Let I_2, I_{311}, I_{321} and I_{322} be as defined by (4.4.1), (4.4.3), (4.4.5) and (4.4.6) respectively. Suppose K and K^* satisfy assumptions (4.1.3) to (4.1.6) with $v = 1, v^* = 1$ and the joint distribution H satisfies assumptions (4.1.11) to (4.1.13). Then for any sequence $\{a_n\}$ of positive numbers with $a_n \rightarrow 0$ and $na_n^4 \rightarrow \infty$, we have

- (i) $\text{var}(n^{1/2} a_n \cdot I_2) \rightarrow 0$ as $n \rightarrow \infty$.
- (ii) $\text{var}(n^{1/2} a_n \cdot I_{311}) \rightarrow 0$ as $n \rightarrow \infty$.
- (iii) $\text{var}(n^{1/2} a_n \cdot I_{321}) \rightarrow 0$ as $n \rightarrow \infty$.
- (iv) $\text{var}(n^{1/2} a_n \cdot I_{322}) \rightarrow 0$ as $n \rightarrow \infty$.

PROOF: (i) We first obtain the following conditional expectations by easy computations:

$$\begin{aligned}
 (4.4.7) \quad & E(G_n(X_1) - G_n(x_0) - G(X_1) + G(x_0) | X_1 = x_1) \\
 &= \frac{1}{n} E \left[1 + \sum_{i=2}^n I(X_{i-1} \leq x_1) - I(X_{i-1} \leq x_0) - \sum_{i=2}^n I(X_{i-1} \leq x_0) - n(G(x_1) - G(x_0)) \right] \\
 &= \frac{1}{n} [I(x_1 > x_0) - (G(x_1) - G(x_0))].
 \end{aligned}$$

$$\begin{aligned}
 (4.4.8) \quad & E[(G_n(X_1) - G_n(x_0) - G(X_1) + G(x_0))^2 | X_1 = x_1] \\
 &= \frac{1}{n^2} E \left\{ \sum_{i=2}^n [I(X_{i-1} \leq x_1) - I(X_{i-1} \leq x_0) - (G(x_1) - G(x_0))] + [I(x_1 > x_0) - (G(x_1) - G(x_0))] \right\}^2 \\
 &= \frac{n-1}{n^2} |G(x_1) - G(x_0)| (1 - |G(x_1) - G(x_0)|) + \frac{1}{n^2} [I(x_1 > x_0) - (G(x_1) - G(x_0))]^2.
 \end{aligned}$$

$$(4.4.9) \quad E[(G_n(X_1) - G_n(x_0) - G(X_1) + G(x_0))(G_n(X_2) - G_n(x_0) - G(X_2) + G(x_0))$$

$$|X_1 = x_1, X_2 = x_2]$$

$$= \frac{1}{n^2} E\left[\left[1 + I(x_2 \leq x_1) + \sum_{i=3}^n I(X_i \leq x_1) - I(x_1 \leq x_0) - I(x_2 \leq x_0) - \sum_{i=3}^n I(X_i \leq x_0)\right.\right.$$

$$\left. - n(G(x_1) - G(x_0))\right]$$

$$\left[1 + I(x_1 \leq x_2) + \sum_{j=3}^n I(X_j \leq x_2) - I(x_1 \leq x_0) - I(x_2 \leq x_0) - \sum_{j=3}^n I(X_j \leq x_0)\right.$$

$$\left. - n(G(x_2) - G(x_0))\right] | X_1 = x_1, X_2 = x_2]$$

$$= \frac{n-2}{n^2} [G(x_1 \wedge x_2) - G(x_1 \wedge x_0) - G(x_2 \wedge x_0) + G(x_0) - (G(x_1) - G(x_0))(G(x_2) - G(x_0))]$$

$$+ \frac{1}{n^2} \sum_{i=1}^2 (I(x_i \leq x_1) - I(x_i \leq x_0) - G(x_1) + G(x_0)) \sum_{j=1}^2 (I(x_j \leq x_2) - I(x_j \leq x_0) - G(x_2) + G(x_0)).$$

From (4.4.1), we have

$$(4.4.10) \quad \text{Var}(n^{1/2} \frac{1}{na_n} I_2) = \frac{1}{na_n^4} [\sum_i \text{Var}(t_i) + \sum_{i \neq j} \text{Cov}(t_i, t_j)]$$

$$= \frac{1}{na_n^4} [n \text{Var}(t_1) + n(n-1) \text{Cov}(t_1, t_2)]$$

where

$$(4.4.11) \quad t_i = [G_n(X_i) - G_n(x_0) - G(X_i) + G(x_0)] K_1 \left(\frac{G(X_i) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_i) - \lambda}{a_n} \right) Y_i.$$

Now

$$(4.4.12) \quad E(t_1) = \iint E[(G_n(x_1) - G_n(x_0) - G(x_1) + G(x_0)) | X_1]$$

$$K_1 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 dH(x_1, y_1)$$

$$= \frac{1}{n} \iint [I(x_1 > x_0) - (G(x_1) - G(x_0))] K_1' \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 dH(x_1, y_1),$$

by (4.4.7)

$$= \frac{1}{n} \iint [I(x_1 > x_0) - (G(x_1) - G(x_0))] K_1' \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 h(x_1, y_1) dx_1 dy_1.$$

Now, using the transformation of variables,

$$(4.4.13) \quad \frac{G(x_1) - G(x_0)}{a_n} = u_1, \quad \frac{F_{x_0}(y_1) - \lambda}{a_n} = v_1$$

and the fact that $P\{G^{-1}(G(X)) = X, F_{x_0}^{-1}(F_{x_0}(Y)) = Y\} = 0$, we can write

$$E(t_1) = \int_{-\lambda/a_n}^{1-\lambda/a_n} \frac{1-G(x_0)}{a_n} \int_{-G(x_0)}^{a_n^2} \frac{1}{n} [I(u_1 > 0) - a_n u_1] K_1'(u_1, v_1) F_{x_0}^{-1}(\lambda + a_n v_1) \\ \frac{h[G^{-1}(G(x_0) + a_n u_1), F_{x_0}^{-1}(\lambda + a_n v_1)]}{g[G^{-1}(G(x_0) + a_n u_1)] f_{x_0}[F_{x_0}^{-1}(\lambda + a_n v_1)]} du_1 dv_1.$$

Since $K(.,.)$ has compact support, say, $[-1, 1] \times [-1, 1]$, without loss of generality, $0 < G(x_0) < 1$, $0 < \lambda < 1$ and $a_n \rightarrow 0$, there exists N' depending on x_0, λ and G only, such that for $n > N'$, the region

$\left(-\frac{G(x_0)}{a_n}, \frac{1-G(x_0)}{a_n}\right) \times \left(\frac{-\lambda}{a_n}, \frac{1-\lambda}{a_n}\right)$ includes the whole support of K in it. Hence,

with the transformation of variables (4.4.13), $E(t_1)$ may be written as

$$(4.4.14) \quad E(t_1) = \frac{a_n^2}{n} \int_{-1}^1 \int_{-1}^1 [I(u_1 > 0) - a_n u_1] K_1'(u_1, v_1) F_{x_0}^{-1}(\lambda + a_n v_1) \\ \frac{h[G^{-1}(G(x_0) + a_n u_1), F_{x_0}^{-1}(\lambda + a_n v_1)]}{g[G^{-1}(G(x_0) + a_n u_1)] f_{x_0}[F_{x_0}^{-1}(\lambda + a_n v_1)]} du_1 dv_1$$

for $n > N'$. Using Taylor expansions as given by lemma (4.1.1) in (4.4.14), which are valid for all $n > N$, we can write for $n > n_0 = \max(N, N')$

$$(4.4.15) \quad E(t_1) = \frac{a_n^2}{n} \int_{-1}^1 \int_{-1}^1 [I(u_1 > 0) - a_n u_1] K_1'(u_1, v_1) [F_{x_0}^{-1}(\lambda) + a_n v_1 D_1 + a_n^2 v_1^2 D_2 + O(a_n^3)] \\ [1 + a_n u_1 D_6 + a_n^2 u_1^2 D_7 + a_n^2 u_1 v_1 D_8 + O(a_n^3)] du_1 dv_1 \\ = O\left(\frac{a_n^2}{n}\right) \text{ as } n \rightarrow \infty.$$

Similarly we have

$$E(t_1^2) = \iint E[(G_n(x_1) - G_n(x_0) - G(x_1))^2 | x_1] K_1'^2\left(\frac{G(x_1) - G(x_0)}{a_n}\right), \frac{F_{x_0}(y_1) - \lambda}{a_n} y_1^2 dH(x_1, y_1) \\ + G(x_0) \\ = \frac{1}{n^2} \iint [(n-1) |G(x_1) - G(x_0)| ((1 - |G(x_1) - G(x_0)|)) + (I(x_1 > x_0) - G(x_1) + G(x_0))^2] \\ K_1'^2\left(\frac{G(x_1) - G(x_0)}{a_n}\right), \frac{F_{x_0}(y_1) - \lambda}{a_n} y_1^2 dH(x_1, y_1), \text{ by (4.4.8).}$$

Upon transformation of variables (4.4.13) and using lemma 4.1.1, we get for $n > n_0$

$$\begin{aligned}
 (4.4.16) \quad E(t_1^2) &= \frac{a_n^2}{n^2} \int_{-1}^1 \int_{-1}^1 [(n-1)|a_n u_1|(1-|a_n u_1|) + (I(u_1 > 0) - a_n u_1)^2] K_1'^2(u_1, v_1) \\
 &\quad [F_{x_0}^{-1}(\lambda) + a_n v_1 D_1 + a_n^2 v_1^2 D_2 + O(a_n^3)] \\
 &\quad [1 + a_n u_1 D_6 + a_n^2 u_1^2 D_7 + a_n^2 u_1 v_1 D_8 + O(a_n^3)] du dv \\
 &= O\left(\frac{a_n^3}{n}\right).
 \end{aligned}$$

Now

$$\begin{aligned}
 (4.4.17) \quad E(t_1 t_2) &= \iiint E[(G_n(x_1) - G_n(x_0) - G(x_1) + G(x_0)) \\
 &\quad (G_n(x_2) - G_n(x_0) - G(x_2) + G(x_0)) | x_1, x_2] \\
 &\quad K_1'\left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n}\right) K_1'\left(\frac{G(x_2) - G(x_0)}{a_n}, \frac{F_{x_0}(y_2) - \lambda}{a_n}\right) \\
 &\quad y_1 y_2 h(x_1, y_1) h(x_2, y_2) dx_1 dy_1 dx_2 dy_2 \\
 &= \frac{n-2}{n^2} \iiint [G(x_1, \lambda x_2) - G(x_1, \lambda x_0) - G(x_2, \lambda x_0) + G(x_0) \\
 &\quad - (G(x_1) - G(x_0))(G(x_2) - G(x_0))] K_1'\left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n}\right) \\
 &\quad K_1'\left(\frac{G(x_2) - G(x_0)}{a_n}, \frac{F_{x_0}(y_2) - \lambda}{a_n}\right) y_1 y_2 h(x_1, y_1) h(x_2, y_2) \\
 &\quad dx_1 dy_1 dx_2 dy_2 \\
 &\quad + O\left(\frac{1}{n^2}\right),
 \end{aligned}$$

on using (4.4.9) and since K_1' is bounded, $E(Y^2) < \infty$. Now, we make the transformation of variables similar to that of (4.4.13), namely,

$$(4.4.18) \quad \frac{G(x_1) - G(x_0)}{a_n} = u_1, \quad \frac{F_{x_0}(y_1) - \lambda}{a_n} = v_1, \quad \frac{G(x_2) - G(x_0)}{a_n} = u_2 \quad \text{and} \\ \frac{F_{x_0}(y_2) - \lambda}{a_n} = v_2.$$

Observe that

$$G(x_1 \wedge x_2) - G(x_1 \wedge x_0) - G(x_2 \wedge x_0) + G(x_0) = \begin{cases} -(G(x_2) - G(x_0)) & \text{if } x_1 < x_2 < x_0 \\ -(G(x_1) - G(x_0)) & \text{if } x_2 < x_1 < x_0 \\ 0 & \text{if } x_1 < x_0 < x_2 \\ 0 & \text{if } x_2 < x_0 < x_1 \\ (G(x_1) - G(x_0)) & \text{if } x_0 < x_1 < x_2 \\ (G(x_2) - G(x_0)) & \text{if } x_0 < x_2 < x_1 \end{cases}$$

so that after the transformation of variables

$$G(x_1 \wedge x_2) - G(x_1 \wedge x_0) - G(x_2 \wedge x_0) + G(x_0) = \begin{cases} -a_n u_2 & \text{if } u_1 < u_2 < 0 \\ -a_n u_1 & \text{if } u_2 < u_1 < 0 \\ 0 & \text{if } u_1 < 0 < u_2 \\ 0 & \text{if } -u_2 < 0 < u_1 \\ a_n u_1 & \text{if } 0 < u_1 < u_2 \\ a_n u_2 & \text{if } 0 < u_2 < u_1 \end{cases}$$

which may be written as

$$G(x_1 \wedge x_2) - G(x_1 \wedge x_0) - G(x_2 \wedge x_0) + G(x_0) = a_n (|u_1| \wedge |u_2|) I(u_1 u_2 > 0).$$

Also, as explained at (4.4.12), after the transformation of variables (4.4.18), the multiple integral in (4.4.17) may be considered as over the set $[-1, 1]^4$, for $n > N'$. Hence we have for $n > N'$

$$\begin{aligned}
E(t_1, t_2) &= \frac{n-2}{n^2} a_n^5 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 [(|u_1| \Lambda |u_2|) I(u_1 u_2 > 0) - a_n^2 u_1 u_2] \\
&\quad K_1'(u_1, v_1) K_1'(u_2, v_2) F_{x_0}^{-1}(\lambda + a_n v_1) F_{x_0}^{-1}(\lambda + a_n v_2) \\
&\quad \frac{h[G^{-1}(G(x_0) + a_n u_1), F_{x_0}^{-1}(\lambda + a_n v_1)]}{g[G^{-1}(G(x_0) + a_n u_1)] f_{x_0}[F_{x_0}^{-1}(\lambda + a_n v_1)]} \\
&\quad \frac{h[G^{-1}(G(x_0) + a_n u_2), F_{x_0}^{-1}(\lambda + a_n v_2)]}{g[G^{-1}(G(x_0) + a_n u_2)] f_{x_0}[F_{x_0}^{-1}(\lambda + a_n v_2)]} \\
&\quad du_1 dv_1 du_2 dv_2.
\end{aligned}$$

Using the Taylor expansions from lemma (4.1.1), we can write, for $n > n_0$

$$\begin{aligned}
(4.4.19) \quad E(t_1, t_2) &= \frac{n-2}{n^2} a_n^5 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 [(|u_1| \Lambda |u_2|) I(u_1 u_2 > 0) - a_n^2 u_1 u_2] K_1'(u_1, v_1) \\
&\quad K_1'(u_2, v_2) \\
&\quad [F_{x_0}^{-1}(\lambda) + a_n v_1 D_1 + a_n^2 v_1^2 D_2 + O(a_n^3)] \\
&\quad [F_{x_0}^{-1}(\lambda) + a_n v_2 D_1 + a_n^2 v_2^2 D_2 + O(a_n^3)] \\
&\quad [1 + a_n u_2 D_6 + a_n^2 u_2^2 D_7 + a_n^2 u_2 v_2 D_8 + O(a_n^3)] \\
&\quad du_1 dv_1 du_2 dv_2 \\
&\quad + O\left(\frac{1}{n^2}\right) \\
&= O\left(\frac{a_n^5}{n}\right) + O\left(\frac{a_n^6}{n^2}\right) + O\left(\frac{1}{n^2}\right).
\end{aligned}$$

Also, from (4.4.15)

$$(4.4.20) \quad E(t_1)E(t_2) = O\left(\frac{a_n^4}{n^2}\right).$$

Now from (4.4.15), (4.4.16), (4.4.19), (4.4.20), and (4.4.10),

assertion (i) of the lemma follows upon observing that $a_n \rightarrow 0$ and $na_n^4 \rightarrow \infty$.

(ii) From (4.4.3), we have

$$(4.4.21) \quad \text{Var}(n^{1/2}a_n I_{311}) = \frac{1}{n^3 a_n^6} [\sum \text{Var}(\ell_1)] = \frac{1}{n^2 a_n^6} \text{Var}(\ell_1),$$

where

$$(4.4.22) \quad \ell_1 = K^* \left(\frac{G(x_1) - G(x_0)}{a_n} \right) K_2' \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) Y_1.$$

Therefore,

$$E(\ell_1^2) = \iint \left[K^* \left(\frac{G(x_1) - G(x_0)}{a_n} \right) \right]^2 K_2'^2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1^2 dH(x_1, y_1),$$

and since K^* and K_2' are bounded and $E(Y_1^2) < \infty$ we have

$$\text{Var}(\ell_1) \leq E(\ell_1^2) < \infty,$$

and assertion (ii) of the lemma now follows from (4.4.21) since $na_n^3 \rightarrow \infty$.

(iii) From (4.4.5) we have

$$(4.4.23) \quad \begin{aligned} \text{Var}(n^{1/2}a_n I_{321}) &= \frac{1}{n^3 a_n^8} \left[\sum_1 \text{Var}(t_{11}) + \sum_{1 \neq j} \text{Cov}(t_{11}, t_{jj}) \right] \\ &= \frac{1}{n^3 a_n^8} [n \text{Var}(t_{11}) + n(n-1) \text{Cov}(t_{11}, t_{22})], \end{aligned}$$

where $t_{11} = [G_n(x_1) - G_n(x_0) - G(x_1) + G(x_0)] K^{**} \left(\frac{G(x_1) - G(x_0)}{a_n} \right)$
 $K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1$.

As in (4.4.16) we find that

$$(4.4.24) \quad E(t_{11}^2) = \iint E[(G_n(x_1) - G_n(x_0) - G(x_1) + G(x_0))^2 | x_1]$$

$$K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right)$$

$$\left[K^{**} \left(\frac{G(x_1) - G(x_0)}{a_n} \right) \right]^2 y_1^2 dH(x_1, y_1)$$

$$= O\left(\frac{a_n^3}{n}\right).$$

Since $\text{Var}(t_{11}) \leq E(t_{11}^2)$

$$|\text{Cov}(t_{11}, t_{22})| \leq [\text{Var}(t_{11}) \cdot \text{Var}(t_{22})]^{1/2} = \text{Var}(t_{11}),$$

and $na_n^4 \rightarrow \infty$, assertion (iii) of the lemma now follows from (4.4.23) and (4.4.24).

(iv) We first obtain the following conditional expectations which are very similar to (4.4.7) to (4.4.9) and whose computations are straightforward:

$$(4.4.25) \quad E[(G_n(x_1) - G_n(x_0) - G(x_1) + G(x_0)) | x_1 = x_1, x_2 = x_2]$$

$$= \frac{1}{n} \sum_{i=1}^2 [I(x_1 \leq x_1) - I(x_1 \leq x_0) - G(x_1) + G(x_0)]$$

$$(4.4.26) \quad E[(G_n(X_1) - G_n(x_0) - G(X_1) + G(x_0)) | X_1 = x_1, X_2 = x_2, X_3 = x_3]$$

$$= \frac{1}{n} \sum_{i=1}^3 [I(x_1 \leq x_i) - I(x_1 \leq x_0) - G(x_1) + G(x_0)]$$

$$(4.4.27) \quad E[(G_n(X_1) - G_n(x_0) - G(X_1) + G(x_0)) | X_1 = x_1, X_2 = x_2, X_3 = x_3, X_4 = x_4]$$

$$= \frac{1}{n} \sum_{i=1}^4 [I(x_1 \leq x_i) - I(x_1 \leq x_0) - G(x_1) + G(x_0)]$$

$$(4.4.28) \quad E[(G_n(X_1) - G_n(x_0) - G(X_1) + G(x_0))^2 | X_1 = x_1, X_2 = x_2]$$

$$= \frac{n-2}{n^2} |G(x_1) - G(x_0)| ((1 - |G(x_1) - G(x_0)|))$$

$$+ \frac{1}{n^2} \left\{ \sum_{i=1}^2 [I(x_1 \leq x_i) - I(x_1 \leq x_0) - G(x_1) + G(x_0)] \right\}^2$$

$$(4.4.29) \quad E[(G_n(X_1) - G_n(x_0) - G(X_1) + G(x_0))^2 | X_1 = x_1, X_2 = x_2, X_3 = x_3]$$

$$= \frac{n-3}{n^2} |G(x_1) - G(x_0)| (1 - |G(x_1) - G(x_0)|)$$

$$+ \frac{1}{n^2} \left\{ \sum_{i=1}^3 [I(x_1 \leq x_i) - I(x_1 \leq x_0) - G(x_1) + G(x_0)] \right\}^2$$

$$\begin{aligned}
(4.4.30) \quad & E[(G_n(X_1) - G_n(x_0) - G(X_1) + G(x_0))(G_n(X_2) - G_n(x_0) - G(X_2) + G(x_0)) \\
& \quad | X_1 = x_1, X_2 = x_2, X_3 = x_3] \\
& = \frac{n-3}{n^2} [G(x_1 \wedge x_2) - G(x_1 \wedge x_0) - G(x_2 \wedge x_0) + G(x_0) \\
& \quad - (G(x_1) - G(x_0))(G(x_2) - G(x_0))] \\
& + \frac{1}{n^2} \sum_{i=1}^3 [I(x_1 \leq x_1) - I(x_1 \leq x_0) - G(x_1) + G(x_0)] \\
& \quad \sum_{j=1}^3 [I(x_j \leq x_2) - I(x_j \leq x_0) - G(x_2) + G(x_0)]
\end{aligned}$$

$$\begin{aligned}
(4.4.31) \quad & E[(G_n(X_1) - G_n(x_0) - G(X_1) + G(x_0)) \\
& \quad (G_n(X_2) - G_n(x_0) - G(X_2) + G(x_0)) | X_1 = x_1, X_2 = x_2, X_3 = x_3, X_4 = x_4] \\
& = \frac{n-4}{n^2} [G(x_1 \wedge x_2) - G(x_1 \wedge x_0) - G(x_2 \wedge x_0) + G(x_0) \\
& \quad - (G(x_1) - G(x_0))(G(x_2) - G(x_0))] \\
& + \frac{1}{n^2} \sum_{i=1}^4 [I(x_1 \leq x_1) - I(x_1 \leq x_0) - G(x_1) + G(x_0)] \\
& \quad \sum_{j=1}^4 [I(x_j \leq x_2) - I(x_j \leq x_0) - G(x_2) + G(x_0)].
\end{aligned}$$

Now, from (4.4.6), we have

$$\text{var}(n^{1/2} a_n I_{322}) = \frac{1}{n^3 a_n^8} \left[\sum_{i \neq j} \text{var}(t_{ij}) + \sum_{\substack{(i,j) \neq (k,l) \\ i \neq j, k \neq l}} \text{cov}(t_{ij}, t_{kl}) \right]$$

where t_{ij} s are given by (4.4.33). We can write

$$(4.4.32) \quad \text{Var}(n^{1/2} a_n I_{322})$$

$$\begin{aligned}
 &= \frac{1}{n^3 a_n^8} \left[\sum_{i \neq j} \text{Var}(t_{ij}) + \sum_{i \neq j \neq k} \text{Cov}(t_{ij}, t_{kj}) + \sum_{i \neq j \neq l} \text{Cov}(t_{ij}, t_{il}) \right. \\
 &\quad + \sum_{i \neq j} \text{Cov}(t_{ij}, t_{ji}) + \sum_{i \neq j \neq l} \text{Cov}(t_{ij}, t_{jl}) + \sum_{i \neq j \neq k} \text{Cov}(t_{ij}, t_{ki}) \\
 &\quad \left. + \sum_{i \neq j \neq k \neq l} \text{Cov}(t_{ij}, t_{kl}) \right] \\
 &= \frac{1}{n^3 a_n^8} \left[n(n-1) \text{Var}(t_{12}) + n(n-1)(n-2) \text{Cov}(t_{12}, t_{13}) \right. \\
 &\quad + n(n-1)(n-2) \text{Cov}(t_{13}, t_{23}) + n(n-1) \text{Cov}(t_{12}, t_{21}) \\
 &\quad + n(n-1)(n-2) \text{Cov}(t_{12}, t_{31}) + n(n-1)(n-2) \text{Cov}(t_{12}, t_{23}) \\
 &\quad \left. + n(n-1)(n-2)(n-3) \text{Cov}(t_{12}, t_{34}) \right] \\
 &= \frac{(n-1)}{n^2 a_n^8} \text{Var}(t_{12}) + \frac{(n-1)(n-2)}{n^2 a_n^8} \text{Cov}(t_{12}, t_{13}) + \frac{(n-1)(n-2)}{n^2 a_n^8} \text{Cov}(t_{13}, t_{23}) \\
 &\quad + \frac{(n-1)}{n^2 a_n^8} \text{Cov}(t_{12}, t_{21}) + \frac{(n-1)(n-2)}{n^2 a_n^8} \text{Cov}(t_{12}, t_{31}) \\
 &\quad + \frac{(n-1)(n-2)}{n^2 a_n^8} \text{Cov}(t_{12}, t_{23}) + \frac{(n-1)(n-2)(n-3)}{n^2 a_n^8} \text{Cov}(t_{12}, t_{34}) \\
 &= A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7,
 \end{aligned}$$

with

$$(4.4.33) \quad t_{1j} = [G_n(x_j) - G_n(x_0) - G(x_j) + G(x_0)] K^{*'} \left(\frac{G(x_j) - G(x_0)}{a_n} \right) \\ K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 I(y_j \leq y_1).$$

Therefore

$$E(t_{12}) = \iiint E[(G_n(x_2) - G(x_0) - G(x_2) + G(x_0)) | x_1, x_2] \\ K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) K^{*'} \left(\frac{G(x_2) - G(x_0)}{a_n} \right) y_1 \\ I(y_2 \leq y_1) dH(x_1, y_1) dH(x_2, y_2). \\ = \frac{1}{n} \iiint \left(\sum_{i=1}^2 [I(x_1 \leq x_i) - I(x_1 \leq x_0) - G(x_1) + G(x_0)] \right) \\ K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) K^{*'} \left(\frac{G(x_2) - G(x_0)}{a_n} \right) \\ y_1 F_{x_2}(y_1) dH(x_1, y_1) dG(x_2), \text{ by (4.4.25).}$$

Now, transformation of variables (4.4.13) and an application of lemma

(4.1.1) yields, for $n > n_0$

$$(4.4.34) \quad E(t_{12}) = \frac{a_n^3}{n} \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \left(\sum_{i=1}^2 [I(u_1 \leq u_i) - I(u_1 \leq 0) - a_n u_1] \right) K_2'(u_1 v_1) K^{*'}(u_2) \\ [F_{x_0}^{-1}(\lambda) + a_n v_1 D_1 + a_n^2 v_1^2 D_2 + O(a_n^3)] \\ [\lambda + a_n u_2 D_3 + a_n v_1 + a_n^2 u_2^2 D_4 + a_n^2 u_2 v_1 D_5 + O(a_n^3)] \\ [1 + a_n u_1 D_6 + a_n u_1^2 D_7 + a_n^2 u_1 v_1 D_8 + O(a_n^3)] du_1 dv_1 du_2$$

$$= O\left(\frac{a_n^3}{n}\right).$$

On the same lines, using (4.4.27) we find that

$$(4.4.35) \quad E(t_{12}^2) = O\left(\frac{a_n^3}{n}\right) + O\left(\frac{a_n^3}{n^2}\right).$$

Hence, we have

$$(4.4.36) \quad \text{var}(t_{12}) = O\left(\frac{a_n^3}{n}\right).$$

Using (4.4.33) we obtain

$$\begin{aligned} (4.4.37) \quad E(t_{12}t_{13}) &= \iiint \iiint E[(G_n(x_2) - G_n(x_0) - G(x_2) + G(x_0)) \\ &\quad (G_n(x_3) - G_n(x_0) - G(x_3) + G(x_0)) | x_1, x_2, x_3] \\ &\quad K_2^2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) K^* \left(\frac{G(x_2) - G(x_0)}{a_n} \right) \\ &\quad K^* \left(\frac{G(x_3) - G(x_0)}{a_n} \right) y_1^2 I(y_2 \leq y_1) I(y_3 \leq y_1) dH(x_1, y_1) \\ &\quad dH(x_2, y_2) dH(x_3, y_3) \\ &= \iiint \iiint \left\{ \frac{n-3}{n^2} [G(x_2 \wedge x_3) - G(x_2 \wedge x_0) - G(x_3 \wedge x_0) + G(x_0)] \right. \\ &\quad \left. - (G(x_2) - G(x_0))(G(x_3) - G(x_0)) \right\} \\ &\quad K^* \left(\frac{G(x_2) - G(x_0)}{a_n} \right) K^* \left(\frac{G(x_3) - G(x_0)}{a_n} \right) \\ &\quad K_2^2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1^2 F_{x_2}(y_1) F_{x_3}(y_1) \\ &\quad dH(x_1, y_1) dG(x_2) dG(x_3) \\ &\quad + O\left(\frac{1}{n^2}\right), \end{aligned}$$

where the last equality follows on using (4.4.30).

Now, by transformation of variables (4.4.18) and an application of lemma (4.1.1) in (4.4.37) yields for $n > n_0$,

$$\begin{aligned}
 (4.4.38) \quad E(t_{12}t_{13}) &= \frac{n-3}{n^2} a_n^5 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 [(|u_2| |u_3|) I(u_2 u_3 > 0) - a_n u_2 u_3] \\
 &\quad K^{*'}(u_2) K^{*'}(u_3) K_2'^2(u_1, v_1) [F_{x_0}^{-1}(\lambda) + a_n v_1 D_1 + a_n^2 v_1^2 D_2 + O(a_n^3)]^2 \\
 &\quad [\lambda + a_n u_2 D_3 + a_n v_1 + a_n^2 u_2^2 D_4 + a_n^2 u_2 v_1 D_5 + O(a_n^3)] \\
 &\quad [\lambda + a_n u_3 D_3 + a_n v_1 + a_n^2 u_3^2 D_4 + a_n^2 u_3 v_1 D_5 + O(a_n^3)] \\
 &\quad [1 + a_n u_1 D_6 + a_n^2 u_1^2 D_7 + a_n^2 u_1 v_1 D_8 + O(a_n^3)] du_1 dv_1 du_3 du_2 \\
 &\quad + O\left(\frac{a_n^4}{n^2}\right) \\
 &= O\left(\frac{a_n^5}{n}\right).
 \end{aligned}$$

Hence, from (4.4.34) and (4.4.38) it follows that

$$(4.4.39) \quad \text{Cov}(t_{12}, t_{13}) = O\left(\frac{a_n^5}{n}\right) + O\left(\frac{a_n^6}{n^2}\right) = O\left(\frac{a_n^5}{n}\right).$$

Proceeding on the same lines, we find that

$$\begin{aligned}
 (4.4.40) \quad \text{Cov}(t_{13}, t_{23}) &= O\left(\frac{a_n^6}{n}\right), \\
 \text{Cov}(t_{12}, t_{21}) &= O\left(\frac{a_n^5}{n}\right), \\
 \text{Cov}(t_{12}, t_{31}) &= O\left(\frac{a_n^6}{n}\right), \text{ and} \\
 \text{Cov}(t_{12}, t_{23}) &= O\left(\frac{a_n^6}{n}\right).
 \end{aligned}$$

Since $a_n \rightarrow 0$ and $na_n^4 \rightarrow \infty$, it follows from (4.4.36), (4.4.39), (4.4.40) and (4.4.32) that each of the terms A_1 to A_6 goes to zero as $n \rightarrow \infty$. So, it suffices to show that $A_7 \rightarrow 0$ as $n \rightarrow \infty$, where we have

$$(4.4.41) \quad A_7 = \frac{(n-1)(n-2)(n-3)}{n^2 a_n^8} [E(t_{12} t_{34}) - E(t_{12})E(t_{34})], \text{ from (4.4.32).}$$

Using (4.4.33) we obtain

$$(4.4.42) \quad E(t_{12} t_{34}) = \iiint \iiint E[(G_n(x_2) - G_n(x_0) - G(x_2) + G(x_0)) \\ (G_n(x_4) - G_n(x_0) - G(x_4) + G(x_0)) | x_1, x_2, x_3, x_4] \\ K^*, \left(\frac{G(x_2) - G(x_0)}{a_n} \right) K^*, \left(\frac{G(x_4) - G(x_0)}{a_n} \right) \\ K_2', \left(\frac{G(x_1) - G(x_0)}{a_n} \right), \frac{F_{x_0}(y_1) - \lambda}{a_n} K_2', \left(\frac{G(x_3) - G(x_0)}{a_n} \right), \frac{F_{x_0}(y_3) - \lambda}{a_n} \\ y_1 y_3 I(y_2 \leq y_1) I(y_4 \leq y_3) dH(x_1, y_1) dH(x_2, y_2) dH(x_3, y_3) dH(x_4, y_4) \\ = \iiint \iiint \left\{ \frac{n-4}{n^2} [G(x_2 \wedge x_4) - G(x_2 \wedge x_0) - G(x_4 \wedge x_0) + G(x_0)] \right. \\ \left. - [G(x_2) - G(x_0)][G(x_4) - G(x_0)] \right\} + o\left(\frac{1}{n^2}\right) \\ K^*, \left(\frac{G(x_2) - G(x_0)}{a_n} \right) K^*, \left(\frac{G(x_4) - G(x_0)}{a_n} \right) K_2', \left(\frac{G(x_1) - G(x_0)}{a_n} \right), \frac{F_{x_0}(y_1) - \lambda}{a_n} \\ K_2', \left(\frac{G(x_3) - G(x_0)}{a_n} \right), \frac{F_{x_0}(y_3) - \lambda}{a_n} y_1 y_3 F_{x_2}(y_1) F_{x_4}(y_3) dH(x_1, y_1) \\ dH(x_3, y_3) dG(x_2) dG(x_4) \right\}$$

where the last equality follows upon using (4.4.31).

Upon transformation of variables (4.4.18) and applying lemma 4.1.1, we have, for $n > n_0$,

$$\begin{aligned}
 (4.4.43) \quad E(t_{12}t_{34}) &= \frac{n-4}{n^2} a_n^6 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 [a_n (|u_2| \wedge |u_4|) I(u_2 u_4 > 0) - a_n^2 u_2 u_4] \\
 &\quad K^{**}(u_2) K^{**}(u_4) K'_2(u_1, v_1) K'_2(u_3, v_3) \\
 &\quad [F_{x_0}^{-1}(\lambda) + a_n v_1 D_1 + a_n^2 v_1^2 D_2 + O(a_n^3)] \\
 &\quad [F_{x_0}^{-1}(\lambda) + a_n v_3 D_1 + a_n^2 v_3^2 D_2 + O(a_n^3)] \\
 &\quad [\lambda + a_n u_2 D_3 + a_n v_1 + a_n^2 u_2^2 D_4 + a_n^2 u_2 v_1 D_5 + O(a_n^3)] \\
 &\quad [\lambda + a_n u_4 D_3 + a_n v_3 + a_n^2 u_4^2 D_4 + a_n^2 u_4 v_3 D_5 + O(a_n^3)] \\
 &\quad [1 + a_n u_1 D_6 + a_n^2 u_1^2 D_7 + a_n^2 u_1 v_1 D_8 + O(a_n^3)] \\
 &\quad + O\left(\frac{a_n^6}{n^2}\right) [1 + a_n u_3 D_6 + a_n^2 u_3^2 D_7 + a_n^2 u_3 v_3 D_8 + O(a_n^3)] du_1 dv_1 du_3 dv_3 du_2 du_4
 \end{aligned}$$

On collecting terms in powers of a_n from (4.4.43) we have

$$\begin{aligned}
 (4.4.44) \quad E(t_{12}t_{34}) &= \frac{n-4}{n^2} a_n^7 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (|u_2| \wedge |u_4|) I(u_2 u_4 > 0) K^{**}(u_2) \\
 &\quad K^{**}(u_4) K'_2(u_1, v_1) K'_2(u_3, v_3) \\
 &\quad \lambda^2 (F_{x_0}^{-1}(\lambda))^2 du_1 dv_1 du_3 dv_3 du_2 du_4 \\
 &\quad + \frac{n-4}{n^2} a_n^8 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (|u_2| \wedge |u_4|) I(u_2 u_4 > 0) K^{**}(u_2) K^{**}(u_4) \\
 &\quad K'_2(u_1, v_1) K'_2(u_3, v_3) [\lambda^2 F_{x_0}^{-1}(\lambda) (v_1 D_1 + v_3 D_1) \\
 &\quad + \lambda (F_{x_0}^{-1}(\lambda))^2 (u_2 D_3 + v_1 + u_4 D_3 + v_3) \\
 &\quad + \lambda^2 (F_{x_0}^{-1}(\lambda))^2 (u_1 D_6 + u_3 D_6)] \\
 &\quad du_1 dv_1 du_3 dv_3 du_2 du_4
 \end{aligned}$$

continued

$$\begin{aligned}
 (4.4.44) \quad & - \frac{n-4}{n^2} a_n^8 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 u_2 u_4 K^{**}(u_2) K^{**}(u_2) K^{**}(u_4) K_2'(u_1, v_1) K_2'(u_3, v_3) \\
 & \lambda^2(F_{x_0}^{-1}(\lambda))^2 du_1 dv_1 du_3 dv_3 du_2 du_4 \\
 & + O\left(\frac{a_n^9}{n}\right) + O\left(\frac{a_n^6}{n}\right) \quad \text{for } n > n_0.
 \end{aligned}$$

Since $\int_{-1}^1 \int_{-1}^1 K_2(u, v) du dv$ is equal to zero, the integrals on the R.H.S. of (4.4.44)

are all equal to zero and it now follows from (4.4.34), (4.4.41) and (4.4.44) that

$$A_7 = O(a_n) + O\left(\frac{1}{na_n^2}\right)$$

which goes to zero as $n \rightarrow \infty$, upon observing that $a_n \rightarrow 0$ and $na_n^2 \rightarrow \infty$.

This proves assertion (iv) of the lemma and hence the proof of the lemma

4.4.1 is complete. $\square$

Next, we shall show that variance of $n^{1/2} a_n (I_{312} - I_{30})$ converges to a finite value.

LEMMA 4.4.2 Under the hypothesis of Lemma 4.4.1, $\text{Var}[n^{1/2} a_n (I_{312} - I_{30})]$ converges to $\sigma_1^2(x_0, \lambda)$, where

$$(4.4.45) \quad \sigma_1^2(x_0, \lambda) = (F_{x_0}^{-1}(\lambda))^2 \int \int K^{*2}(u) K_2^2(v) du dv$$

$$K_2(v) = \int K(u, v) du$$

and I_{312}, I_{30} are given by (4.4.44) and (4.4.1) respectively.

PROOF: We shall write

$$\begin{aligned}
 (4.4.46) \quad \text{Var}[n^{1/2} a_n (I_{312} - I_{30})] &= \text{Var}(n^{1/2} a_n I_{312}) + \text{Var}(n^{1/2} a_n I_{30}) \\
 &\quad - 2 \text{Cov}(n^{1/2} a_n I_{312}, n^{1/2} a_n I_{30}).
 \end{aligned}$$

From (4.4.4) we have

$$\text{Var}(n^{1/2} a_n I_{312}) = \frac{1}{n^3 a_n^6} \left[\sum_{i \neq j} \text{Var}(l_{ij}) + \sum_{\substack{(i,j) \neq (k,m) \\ i \neq j \quad k \neq m}} \text{Cov}(l_{ij}, l_{km}) \right]$$

where

$$(4.4.47) \quad l_{ij} = K^* \left(\frac{G(X_j) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(X_i) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_i) - \lambda}{a_n} \right) Y_i I(Y_j \leq Y_i).$$

As in (4.4.32), we can write

$$\begin{aligned} (4.4.48) \quad \text{Var}(n^{1/2} a_n I_{312}) &= \frac{1}{n^3 a_n^6} [n(n-1) \text{Var}(l_{12}) + n(n-1)(n-2) \text{Cov}(l_{12}, l_{13}) \\ &\quad + n(n-1)(n-2) \text{Cov}(l_{13}, l_{23}) + n(n-1) \text{Cov}(l_{12}, l_{21}) \\ &\quad + n(n-1)(n-2) \text{Cov}(l_{12}, l_{31}) + n(n-1)(n-2) \text{Cov}(l_{12}, l_{23}) \\ &\quad + n(n-1)(n-2)(n-3) \text{Cov}(l_{12}, l_{34})] \\ &= \frac{n-1}{n^2 a_n^6} \text{Var}(l_{12}) + \frac{(n-1)(n-2)}{n^2 a_n^6} \text{Cov}(l_{12}, l_{13}) + \frac{(n-1)(n-2)}{n^2 a_n^6} \text{Cov}(l_{13}, l_{23}) \\ &\quad + \frac{(n-1)}{n^2 a_n^6} \text{Cov}(l_{12}, l_{21}) + \frac{(n-1)(n-2)}{n^2 a_n^6} \text{Cov}(l_{12}, l_{31}) \\ &\quad + \frac{(n-1)(n-2)}{n^2 a_n^6} \text{Cov}(l_{12}, l_{23}) + \frac{(n-1)(n-2)(n-3)}{n^2 a_n^6} \text{Cov}(l_{12}, l_{34}) \\ &= B_1 + B_2 + B_3 + B_4 + B_5 + B_6 + B_7. \end{aligned}$$

First, observe that $\text{Cov}(l_{12}, l_{34})$ is zero, since l_{12} and l_{34} are functions of two independent sets of rv's. Hence,

$$(4.4.49) \quad B_7 = 0.$$

Now, using (4.4.47)

$$\begin{aligned}
 E(l_{12}^2) &= \iiint K^* \left(\frac{G(x_2) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(x_1) - G(x_0)}{a_n} \right) \frac{F_{x_0}(y_1) - \lambda}{a_n} y_1^2 I(y_2 \leq y_1) \\
 &\quad dH(x_1, y_1) dH(x_2, y_2) \\
 &= \iiint K^* \left(\frac{G(x_2) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(x_1) - G(x_0)}{a_n} \right) \frac{F_{x_0}(y_1) - \lambda}{a_n} y_1^2 F_{x_2}(y_1) \\
 &\quad dH(x_1, y_1) dG(x_2).
 \end{aligned}$$

Upon transformation of variables (4.4.18) and using lemma 4.1.1, we easily see that

$$(4.4.50) \quad E(l_{12}^2) = O(a_n^3) \quad \text{and hence} \quad \text{Var}(l_{12}) = O(a_n^3).$$

Consequently it follows from (4.4.48) that

$$(4.4.51) \quad B_1 = O\left(\frac{1}{na_n^3}\right), \quad \text{which goes to zero as } n \rightarrow \infty.$$

Since $\text{Var}(l_{12}) = \text{Var}(l_{21})$, by Schwartz's inequality, we have $|\text{Cov}(l_{12}, l_{21})| \leq \text{Var}(l_{12})$ and it follows from (4.4.48) and (4.4.50) that

$$(4.4.52) \quad B_4 = O\left(\frac{1}{na_n^3}\right), \quad \text{which again goes to zero as } n \rightarrow \infty.$$

Hence, we now need to compute the terms B_2, B_3, B_5 and B_6 only. We have from (4.4.48)

$$(4.4.53) \quad B_2 = \frac{(n-1)(n-2)}{n^2 a_n^6} [E(l_{12} l_{13}) - E(l_{12})E(l_{13})].$$

We obtain

$$(4.4.54) \quad E(l_{12})E(l_{13})$$

$$\begin{aligned}
 &= \iiint \iiint K^* \left(\frac{G(x_2) - G(x_0)}{a_n} \right) K^* \left(\frac{G(x_4) - G(x_0)}{a_n} \right) \\
 &\quad K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) K_2 \left(\frac{G(x_3) - G(x_0)}{a_n}, \frac{F_{x_0}(y_3) - \lambda}{a_n} \right) \\
 &\quad y_1 y_3 I(y_2 \leq y_1) I(y_4 \leq y_3) dH(x_1, y_1) dH(x_2, y_2) dH(x_3, y_3) dH(x_4, y_4) \\
 &= \iiint \iiint K^* \left(\frac{G(x_2) - G(x_0)}{a_n} \right) K^* \left(\frac{G(x_4) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) \\
 &\quad K_2 \left(\frac{G(x_3) - G(x_0)}{a_n}, \frac{F_{x_0}(y_3) - \lambda}{a_n} \right) y_1 y_3 F_{x_2}(y_1) F_{x_4}(y_3) dH(x_1, y_1) \\
 &\quad dH(x_3, y_3) dG(x_2) dG(x_4).
 \end{aligned}$$

Upon usual transformation of variables as in (4.4.18) and applying lemma 4.1.1, and collecting terms up to order a_n^6 , we get, for $n > n_0$,

$$\begin{aligned}
 E(l_{12})E(l_{13}) &= a_n^6 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 K^*(u_2) K^*(u_4) K_2'(u_1, v_1) K_2'(u_3, v_3) \lambda^2 (F_{x_0}^{-1}(\lambda))^2 \\
 &\quad du_1 dv_1 du_3 dv_3 du_2 du_4 \\
 &\quad + o(a_n^7) \\
 &= a_n^6 \lambda^2 (F_{x_0}^{-1}(\lambda))^2 \left(\int_{-1}^1 \int_{-1}^1 K^*(u_2) K^*(u_4) du_2 du_4 \right) \left(\int_{-1}^1 \int_{-1}^1 K_2'(u_1, v_1) du_1 dv_1 \right)^2 \\
 &\quad + o(a_n^7) \\
 &= o(a_n^7),
 \end{aligned}$$

$$\text{since } \int_{-1}^1 \int_{-1}^1 K_2'(u_1, v_1) du_1 dv_1 = 0.$$

From (4.4.47), we write

$$\begin{aligned}
 (4.4.56) \quad E(l_{12}l_{13}) &= \iiint \tilde{K}_2^2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) K^* \left(\frac{G(x_2) - G(x_0)}{a_n} \right) \\
 &\quad K^* \left(\frac{G(x_3) - G(x_0)}{a_n} \right) y_1^2 I(y_2 \leq y_1) I(y_3 \leq y_1) \\
 &\quad dH(x_1, y_1) dH(x_2, y_2) dH(x_3, y_3) \\
 &= \iiint K_2^2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) K^* \left(\frac{G(x_2) - G(x_0)}{a_n} \right) \\
 &\quad K^* \left(\frac{G(x_3) - G(x_0)}{a_n} \right) y_1^2 F_{x_2}(y_1) F_{x_3}(y_1) dH(x_1, y_1) \\
 &\quad dG(x_2) dG(x_3).
 \end{aligned}$$

By transformation of variables (4.4.18) and using lemma 4.1.1 we get, for

$n > n_0$,

$$\begin{aligned}
 E(l_{12}l_{13}) &= a_n^4 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 K^*(u_2) K^*(u_3) K_2^2(u_1, v_1) [F_{x_0}^{-1}(\lambda) + a_n v_1 D_1 + a_n^2 v_1^2 D_2 + O(a_n^3)]^2 \\
 &\quad [\lambda + a_n u_2 D_3 + a_n v_1 + a_n^2 u_2^2 D_4 + a_n^2 u_2 v_1 D_5 + O(a_n^3)] \\
 &\quad [\lambda + a_n u_3 D_3 + a_n v_1 + a_n^2 u_3^2 D_4 + a_n^2 u_3 v_1 D_5 + O(a_n^3)] \\
 &\quad [1 + a_n u_1 D_6 + a_n^2 u_1^2 D_7 + a_n^2 u_1 v_1 D_8 + O(a_n^3)] du_1 dv_1 du_2 du_3.
 \end{aligned}$$

On collecting terms in powers of a_n from the R.H.S. integral while using the hypothesis $\int K^*(u) du = 1$, $\int u K^*(u) du = 0$ and using the resulting expression along with (4.4.54) in (4.4.53) we obtain

$$\begin{aligned}
(4.4.58) \quad B_2 = & \frac{(n-1)(n-2)}{n^2 a_n^2} \iint K_2'^2(u_1, v_1) \lambda^2 (F_{x_0}^{-1}(\lambda))^2 du_1 dv_1 \\
& + \frac{(n-1)(n-2)}{n^2 a_n^2} \iint K_2'^2(u_1, v_1) [2\lambda^2 F_{x_0}^{-1}(\lambda) v_1 D_1 + 2\lambda (F_{x_0}^{-1}(\lambda))^2 v_1 \\
& \quad + \lambda^2 (F_{x_0}^{-1}(\lambda))^2 u_1 D_6] du_1 dv_1 \\
& + \frac{(n-1)(n-2)}{n^2} \iint K_2'^2(u_1, v_1) [2\lambda^2 F_{x_0}^{-1}(\lambda) v_1^2 D_2 + \lambda^2 v_1^2 D_1^2 \\
& \quad + \lambda^2 (F_{x_0}^{-1}(\lambda))^2 (u_1^2 D_7 + u_1 v_1 D_8) + 4\lambda F_{x_0}^{-1}(\lambda) v_1^2 D_1 + 2\lambda^2 F_{x_0}^{-1}(\lambda) v_1 u_1 D_1 D_6 \\
& \quad + (F_{x_0}^{-1}(\lambda))^2 v_1^2 + 2\lambda (F_{x_0}^{-1}(\lambda))^2 v_1 u_1 D_6] du_1 dv_1 \\
& + \frac{(n-1)(n-2)}{n^2} \iint K_2'^2(u_1, v_1) 2(F_{x_0}^{-1}(\lambda))^2 D_4 du_1 dv_1 \left(\int u_2^2 K^*(u_2) du_2 \right) \\
& + O(a_n).
\end{aligned}$$

Now, $\text{Cov}(l_{13}, l_{23}) = E(l_{13} l_{23}) - E(l_{13}) E(l_{23})$ and

$$\begin{aligned}
(4.4.59) \quad E(l_{13} l_{23}) = & \iiint K^2 \left(\frac{G(x_3) - G(x_0)}{a_n} \right) K_2' \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) \\
& K_2' \left(\frac{G(x_2) - G(x_0)}{a_n}, \frac{F_{x_0}(y_2) - \lambda}{a_n} \right) y_1 y_2 I(y_3 \leq y_1 \wedge y_2) \\
& dH(x_1, y_1) dH(x_2, y_2) dH(x_3, y_3) \\
= & 2 \iiint K^2 \left(\frac{G(x_3) - G(x_0)}{a_n} \right) K_2' \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) \\
& K_2' \left(\frac{G(x_2) - G(x_0)}{a_n}, \frac{F_{x_0}(y_2) - \lambda}{a_n} \right) y_1 y_2 I(y_3 \leq y_1) I(y_1 \leq y_2) \\
& dH(x_1, y_1) dH(x_2, y_2) dH(x_3, y_3).
\end{aligned}$$

After transformation of variables as in (4.4.18) and using Taylor expansions from lemma 4.1.1, we get, for $n > n_0$

$$(4.4.60) \quad E(l_{13}l_{23}) = 2a_n^5 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \{K^*(u_3)\}^2 K'_2(u_1, v_1) K'_2(u_2, v_2) I(v_1 \leq v_2)$$

$$[F_{x_0}^{-1}(\lambda) + a_n v_1 D_1 + a_n^2 v_1^2 D_2 + O(a_n^3)]$$

$$[F_{x_0}^{-1}(\lambda) + a_n v_2 D_1 + a_n^2 v_2^2 D_2 + O(a_n^3)]$$

$$[\lambda + a_n u_3 D_3 + a_n \bar{v}_1 + a_n^2 u_3^2 D_4 + a_n^2 u_3 v_1 D_5 + O(a_n^3)]$$

$$[1 + a_n u_1 D_6 + a_n^2 u_1^2 D_7 + a_n^2 u_1 v_1 D_8 + O(a_n^3)]$$

$$[1 + a_n u_2 D_6 + a_n^2 u_2^2 D_7 + a_n^2 u_2 v_2 D_8 + O(a_n^3)]$$

$$du_1 dv_1 du_2 dv_2 du_3$$

$$= 2a_n^5 \lambda (F_{x_0}^{-1}(\lambda))^2 \left(\int_{\{v_1 \leq v_2\}} K'_2(u_1, v_1) K'_2(u_2, v_2) du_1 dv_1 du_2 dv_2 \right)$$

$$\int \{K^*(u_3)\}^2 du_3$$

$$+ 2a_n^6 \int_{\{v_1 \leq v_2\}} \{K^*(u_3)\}^2 K'_2(u_1, v_1) K'_2(u_2, v_2)$$

$$[\lambda(v_1 + v_2) D_1 + (F_{x_0}^{-1}(\lambda))^2 (u_3 D_3 + v_1)$$

$$+ \lambda (F_{x_0}^{-1}(\lambda))^2 (u_1 + u_2) D_6]$$

$$+ O(a_n^7).$$

Since $\iint K'_2(u, v) du dv = 0$, we observe that

$$(4.4.61) \quad 2 \iiint_{\{v_1 \leq v_2\}} K'_2(u_1, v_1) K'_2(u_2, v_2) du_1 dv_1 du_2 dv_2 = \iiint K'_2(u_1, v_1) K'_2(u_2, v_2) du_1 dv_1 du_2 dv_2$$

$$= (\iint K_2(u_1, v_1) du_1 dv_1)^2 = 0,$$

$$(4.4.62) \quad 2 \iiint_{\{v_1 \leq v_2\}} (v_1 + v_2) K'_2(u_1, v_2) K'_2(u_2, v_2) du_1 dv_1 du_2 dv_2 = \iiint (v_1 + v_2) K'_2(u_1, v_1) K'_2(u_2, v_2) du_1 dv_1 du_2 dv_2$$

$$= 0,$$

$$(4.4.63) \quad 2 \iiint_{\{v_1 \leq v_2\}} (u_1 + u_2) K'_2(u_1, v_1) K'_2(u_2, v_2) dv_1 du_1 du_2 dv_2 = \iiint (u_1 + u_2) K'_2(u_1, v_1) K'_2(u_2, v_2) du_1 dv_1 du_2 dv_2$$

$$= 0.$$

Hence, from (4.4.54) and (4.4.60) to (4.4.63), we have

$$(4.4.64) \quad \text{Cov}(l_{13}, l_{23}) = 2(F_{x_0}^{-1}(\lambda))^2 a_n^6 \iiint_{\{v_1 \leq v_2\}} v_1 K'_2(u_1, v_1) K'_2(u_2, v_2) (\int |K^*(u_3)|^2 du_3) du_1 dv_1 du_2 dv_2$$

$$+ o(a_n^7).$$

Therefore, from (4.4.48), it follows that

$$(4.4.65) \quad B_3 = \frac{(n-1)(n-2)}{n^2} 2(F_{x_0}^{-1}(\lambda))^2 \iiint_{\{v_1 \leq v_2\}} v_1 K'_2(u_1, v_1) K'_2(u_2, v_2) (\int |K^*(u_3)|^2 du_3) du_1 dv_1 du_2 dv_2$$

$$+ o(a_n).$$

From (4.4.47) we have

$$\begin{aligned}
 (4.4.66) \quad E(l_{12}l_{31}) &= \iiint \iiint K^* \left(\frac{G(x_2) - G(x_0)}{a_n} \right) K^* \left(\frac{G(x_1) - G(x_0)}{a_n} \right) \\
 &\quad K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) \\
 &\quad K_2 \left(\frac{G(x_3) - G(x_0)}{a_n}, \frac{F_{x_0}(y_3) - \lambda}{a_n} \right) y_1 y_3 I(y_2 \leq y_1) I(y_1 \leq y_3) \\
 &\quad dH(x_1, y_1) dH(x_2, y_2) dH(x_3, y_3) \\
 &= \iiint \iiint K^* \left(\frac{G(x_2) - G(x_0)}{a_n} \right) K^* \left(\frac{G(x_1) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) \\
 &\quad K_2 \left(\frac{G(x_3) - G(x_0)}{a_n}, \frac{F_{x_0}(y_3) - \lambda}{a_n} \right) y_1 y_3 I(y_1 \leq y_3) F_{x_2}(y_1) dH(x_1, y_1) \\
 &\quad dH(x_3, y_3) dG(x_2),
 \end{aligned}$$

and note that

$$(4.4.67) \quad E(l_{12}l_{31}) = E(l_{12}l_{23}).$$

Again, transformation of variables as in (4.4.18) and an application of lemma 4.1.1 in (4.4.66) yields, for $n > n_0$

$$\begin{aligned}
 E(l_{12}l_{31}) &= a_n^5 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 K^*(u_2) K^*(u_1) K_2'(u_1, v_1) K_2'(u_3, v_3) I(v_1 \leq v_3) \\
 &\quad [F_{x_0}^{-1}(\lambda) + a_n v_1 D_1 + a_n^2 v_1^2 D_2 + O(a_n^3)] \\
 &\quad [F_{x_0}^{-1}(\lambda) + a_n v_3 D_1 + a_n^2 v_3^2 D_2 + O(a_n^3)] \\
 &\quad [\lambda + a_n u_2 D_1 + a_n v_1 + a_n^2 u_2^2 D_1 + a_n^2 u_2 v_1 D_2 + O(a_n^3)]
 \end{aligned}$$

$$[1 + a_n u_1 D_6 + a_n^2 u_1^2 D_7 + a_n^2 u_1 v_1 D_8 + o(a_n^3)]$$

$$[1 + a_n u_3 D_6 + a_n^2 u_3^2 D_7 + a_n^2 u_3 v_3 D_8 + o(a_n^3)]$$

$$du_1 dv_1 du_3 dv_3 du_2,$$

which we may write as

$$E(\ell_{12} \ell_{31}) = a_n^5 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \lambda (F_{x_0}^{-1}(\lambda))^2 K^*(u_2) K^*(u_1) K_2'(u_1, v_1) K_2'(u_3, v_3) I(v_1 \leq v_3)$$

$$du_1 dv_1 du_3 dv_3 du_2$$

$$+ a_n^6 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 K^*(u_2) K^*(u_1) K_2'(u_1, v_1) K_2'(u_3, v_3) I(v_1 \leq v_3)$$

$$[\lambda F_{x_0}^{-1}(\lambda)(v_1 + v_3) D_1 + (F_{x_0}^{-1}(\lambda))^2 (u_2 D_3 + v_1)$$

$$+ \lambda (F_{x_0}^{-1}(\lambda))^2 (u_1 + u_3) D_6] du_1 dv_1 du_3 dv_3 du_2$$

$$+ o(a_n^7).$$

Since $\int_{-1}^1 K^*(u) du = 1$, $\int_{-1}^1 u K^*(u) du = 0$, by hypothesis, the above expression reduces to

$$(4.4.68) \quad E(\ell_{12} \ell_{31}) = a_n^5 \iiint \int K^*(u_1) K_2'(u_1, v_1) K_2'(u_3, v_3) I(v_1 \leq v_3) \lambda (F_{x_0}^{-1}(\lambda))^2$$

$$du_1 dv_1 du_3 dv_3$$

$$+ a_n^6 \iiint \int K^*(u_1) K_2'(u_1, v_1) K_2'(u_3, v_3) I(v_1 \leq v_3)$$

$$[\lambda F_{x_0}^{-1}(\lambda)(v_1 + v_3) D_1 + (F_{x_0}^{-1}(\lambda))^2 v_1 + \lambda (F_{x_0}^{-1}(\lambda))^2$$

$$(u_1 + u_3) D_6] du_1 dv_1 du_3 dv_3$$

$$+ o(a_n^7),$$

$$\text{for } n > n_0.$$

Hence, it follows from (4.4.48), (4.4.54) and (4.4.67) that

$$\begin{aligned}
 (4.4.69) \quad B_5 + B_6 &= \frac{(n-1)(n-2)}{n^2 a_n} \iiint K^*(u_1) K'_2(u_1, v_1) K'_2(u_3, v_3) I(v_1 \leq v_3) \\
 &\quad 2\lambda (F_{x_0}^{-1}(\lambda))^2 du_1 dv_1 du_3 dv_3 \\
 &\quad + \frac{(n-1)(n-2)}{n^2} \iiint K^*(u_1) K'_2(u_1, v_1) K'_2(u_3, v_3) I(v_1 \leq v_3) \\
 &\quad [2\lambda F_{x_0}^{-1}(\lambda)(v_1 + v_3) D_1 + 2(F_{x_0}^{-1}(\lambda))^2 v_1 \\
 &\quad + 2\lambda (F_{x_0}^{-1}(\lambda))^2 (u_1 + u_3) D_6] du_1 dv_1 du_3 dv_3 \\
 &\quad + o(a_n).
 \end{aligned}$$

Therefore

$$\begin{aligned}
 (4.4.70) \quad \text{Var}(n^{1/2} a_n I_{312}) &= \frac{(n-1)(n-2)}{n^2 a_n^2} \int K_2'^2(u_1, v_1) \lambda^2 (F_{x_0}^{-1}(\lambda))^2 du_1 dv_1 \\
 &\quad + \frac{(n-1)(n-2)}{n^2 a_n} \int K_2'^2(u_1, v_1) [2\lambda^2 F_{x_0}^{-1}(\lambda) v_1 D_1 + 2\lambda (F_{x_0}^{-1}(\lambda))^2 v_1 + \lambda^2 (F_{x_0}^{-1}(\lambda)) u_1 D_6] du_1 dv_1 \\
 &\quad + \frac{(n-1)(n-2)}{n^2} \int K_2'^2(u_1, v_1) [2\lambda^2 F_{x_0}^{-1}(\lambda) v_1^2 D_2 + \lambda^2 v_1^2 D_1^2 + \lambda^2 (F_{x_0}^{-1}(\lambda))^2 (u_1^2 D_7 + u_1 v_1 D_8) \\
 &\quad + 4\lambda F_{x_0}^{-1}(\lambda) v_1^2 D_1 + 2\lambda^2 F_{x_0}^{-1}(\lambda) v_1 u_1 D_1 D_6 + (F_{x_0}^{-1}(\lambda))^2 v_1^2 + 2\lambda (F_{x_0}^{-1}(\lambda))^2 \\
 &\quad v_1 u_1 D_6] du_1 dv_1
 \end{aligned}$$

continued

$$\begin{aligned}
& + \frac{(n-1)(n-2)}{n^2} \int \int K_2'^2(u_1, v_1) 2\lambda(F_{x_0}^{-1}(\lambda))^2 D_4 du_1 dv_1 (\int u_2^2 K^*(u_2) du_2) \\
& + \frac{(n-1)(n-2)}{n^2} 2(F_{x_0}^{-1}(\lambda))^2 \int \int \int v_1 I(v_1 \leq v_2) K_2'(u_1, v_1) K_2'(u_2, v_2) du_1 dv_1 du_2 dv_2 \\
& \quad (\int \{K^*(u_3)\}^2 du_3) \\
& + \frac{(n-1)(n-2)}{n^2 a_n} 2\lambda(F_{x_0}^{-1}(\lambda))^2 \int \int \int K^*(u_1) K_2'(u_1, v_1) K_2'(u_3, v_3) I(v_1 \leq v_3) du_1 dv_1 du_3 dv_3 \\
& + \frac{(n-1)(n-2)}{n^2} \int \int \int K^*(u_1) K_2'(u_1, v_1) K_2'(u_3, v_3) I(v_1 \leq v_3) \\
& \quad [2\lambda F_{x_0}^{-1}(\lambda)(v_1 + v_3) D_1 + 2(F_{x_0}^{-1}(\lambda))^2 v_1 + 2\lambda(F_{x_0}^{-1}(\lambda))^2 (u_1 + u_3) D_6] \\
& \quad du_1 dv_1 du_3 dv_3 \\
& + O\left(\frac{1}{na_n^3}\right) + O(a_n).
\end{aligned}$$

Now, from (4.4.1) we have

$$I_{30} = \frac{1}{na_n^3} \sum_{i=1}^n F_{x_0}(Y_i) K_2'\left(\frac{G(X_i) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_i) - \lambda}{a_n}\right) Y_i, \text{ and hence}$$

$$\begin{aligned}
(4.4.71) \quad \text{Var}(n^{1/2} a_n I_{30}) &= \frac{na_n^2}{n^2 a_n^6} [\sum_i \text{Var}(p_i)] \\
&= \frac{1}{a_n} \text{Var}(p_1),
\end{aligned}$$

where

$$(4.4.72) \quad p_i = K_2'\left(\frac{G(X_i) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_i) - \lambda}{a_n}\right) Y_i F_{x_0}(Y_i).$$

Therefore

$$E(p_1^2) = \int \int K_2'^2\left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n}\right) y_1^2 F_{x_0}^2(y_1) dH(x_1, y_1).$$

$$\begin{aligned}
(4.4.74) \quad [E(p_1)]^2 &= a_n^4 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 K_2'(u_1, v_1) K_2'(u_2, v_2) [F_{x_0}^{-1}(\lambda) + a_n v_1 D_1 + a_n^2 v_1^2 D_2 + O(a_n^3)] \\
&\quad [F_{x_0}^{-1}(\lambda) + a_n v_2 D_1 + a_n^2 v_2^2 D_2 + O(a_n^3)] [\lambda + a_n v_1] \\
&\quad [\lambda + a_n v_2] [1 + a_n u_1 D_6 + a_n^2 u_1^2 D_7 + a_n^2 u_1 v_1 D_8 + O(a_n^3)] \\
&\quad [1 + a_n u_2 D_6 + a_n^2 u_2^2 D_7 + a_n^2 u_2 v_2 D_8 + O(a_n^3)] du_1 dv_1 du_2 dv_2 \\
&= a_n^4 \lambda^2 (F_{x_0}^{-1}(\lambda))^2 \int \int \int \int K_2'(u_1, v_1) K_2'(u_2, v_2) du_1 dv_1 du_2 dv_2 \\
&\quad + O(a_n^5) \\
&= O(a_n^5), \text{ since } \int \int K_2'(u, v) du dv = 0.
\end{aligned}$$

Hence, from (4.4.71), (4.4.73) and (4.4.74) we have, for $n > n_0$,

$$\begin{aligned}
(4.4.75) \quad \text{Var}(n^{1/2} a_n I_{30}) &= \frac{1}{a_n^2} \int \int K_2'^2(u_1, v_1) \lambda^2 (F_{x_0}^{-1}(\lambda))^2 du_1 dv_1 \\
&\quad + \frac{1}{a_n} \int \int K_2'^2(u_1, v_1) [2\lambda^2 F_{x_0}^{-1}(\lambda) v_1 D_1 + 2\lambda (F_{x_0}^{-1}(\lambda))^2 v_1 \\
&\quad + \lambda^2 (F_{x_0}^{-1}(\lambda))^2 u_1 D_6] du_1 dv_1 \\
&\quad + \int \int K_2'^2(u_1, v_1) [2\lambda^2 F_{x_0}^{-1}(\lambda) v_1^2 D_2 + \lambda^2 v_1^2 D_1^2 + (F_{x_0}^{-1}(\lambda))^2 v_1^2 \\
&\quad + \lambda^2 (F_{x_0}^{-1}(\lambda))^2 (u_1^2 D_7 + u_1 v_1 D_8) + 4\lambda F_{x_0}^{-1}(\lambda) v_1^2 D_1 \\
&\quad + 2\lambda^2 F_{x_0}^{-1}(\lambda) v_1 u_1 D_1 D_6 + 2\lambda (F_{x_0}^{-1}(\lambda))^2 v_1 u_1 D_6] \\
&\quad + O(a_n) .
\end{aligned}$$

Now, observe that

$$\begin{aligned}
 (4.4.76) \quad \text{Cov}(n^{1/2} a_n I_{312}, n^{1/2} a_n I_{30}) &= n a_n^2 \text{Cov}\left(\frac{1}{n^2 a_n} \sum_{i \neq j} \sum l_{ij}, \frac{1}{n a_n} \sum p_m\right) \\
 &= \frac{1}{n^2 a_n^5} \sum_{i \neq j} \sum_m \text{Cov}(l_{ij}, p_m) \\
 &= \frac{1}{n^2 a_n^5} \left[\sum_{i \neq j} \text{Cov}(l_{ij}, p_1) + \sum_{i \neq j} \text{Cov}(l_{ij}, p_j) \right] \\
 &= \frac{1}{n^2 a_n^5} [n(n-1) \text{Cov}(l_{12}, p_1) + n(n-1) \text{Cov}(l_{21}, p_1)] \\
 &= \frac{n-1}{n a_n^5} \text{Cov}(l_{12}, p_1) + \frac{n-1}{n a_n^5} \text{Cov}(l_{21}, p_1)
 \end{aligned}$$

where l_{ij} 's are defined by (4.4.47) and p_m 's by (4.4.72). We shall now compute the covariances that appear in the last expression:

$$\begin{aligned}
 (4.4.77) \quad E(l_{12} p_1) &= E\left[K^* \left(\frac{G(x_2) - G(x_0)}{a_n}\right) K_2' \left(\frac{2G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n}\right) y_1^2 F_{x_0}(y_1) I(y_2 \leq y_1)\right] \\
 &= \iiint K^* \left(\frac{G(x_2) - G(x_0)}{a_n}\right) K_2' \left(\frac{2G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n}\right) y_1^2 F_{x_0}(y_1) I(y_2 \leq y_1) \\
 &\quad dH(x_1, y_1) dH(x_2, y_2) \\
 &= \iiint K^* \left(\frac{G(x_2) - G(x_0)}{a_n}\right) K_2' \left(\frac{2G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n}\right) \\
 &\quad y_1^2 F_{x_0}(y_1) F_{x_2}(y_1) dH(x_1, y_1) dG(x_2).
 \end{aligned}$$

By the transformation of variables (4.4.18) together with lemma 4.1.1 yields for $n > n_0$

$$E(\ell_{12} p_1) = a_n^3 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 K^*(u_2) K_2'^2(u_1, v_1) [F_{x_0}^{-1}(\lambda) + a_n v_1 D_1 a_n^2 v_1^2 D_2 + O(a_n^3)]^2 \\ [\lambda + a_n v_1] [\lambda + a_n u_2 D_3 + a_n v_1 + a_n^2 u_2^2 D_4 + a_n^2 u_2 v_1 D_5 + O(a_n^3)] \\ [1 + a_n u_1 D_6 + a_n^2 u_1^2 D_7 + a_n^2 u_1 v_1 D_8 + O(a_n^3)] du_1 dv_1 du_2.$$

Considering the terms up to order a_n^5 and recalling that $\int K^*(u) du = 1$, $\int u K^*(u) du = 0$, we have

$$(4.4.78) \quad E(\ell_{12} p_1) = a_n^3 \iint K_2'^2(u_1, v_1) \lambda^2 (F_{x_0}^{-1}(\lambda))^2 du_1 dv_1 \\ + a_n^4 \iint K_2'^2(u_1, v_1) [2\lambda^2 F_{x_0}^{-1}(\lambda) v_1 D_1 + 2\lambda (F_{x_0}^{-1}(\lambda))^2 v_1 + \lambda^2 (F_{x_0}^{-1}(\lambda))^2 \\ u_1 D_6] du_1 dv_1 \\ + a_n^5 \iint K_2'^2(u_1, v_1) [2\lambda^2 F_{x_0}^{-1}(\lambda) v_1^2 D_2 + \lambda^2 v_1^2 D_1^2 + \lambda^2 (F_{x_0}^{-1}(\lambda))^2 (u_1^2 D_7 \\ + u_1 v_1 D_8) + 4\lambda F_{x_0}^{-1}(\lambda) v_1^2 D_1 + 2\lambda^2 F_{x_0}^{-1}(\lambda) v_1 u_1 D_1 D_6 \\ + (F_{x_0}^{-1}(\lambda))^2 v_1^2 + 2\lambda (F_{x_0}^{-1}(\lambda))^2 v_1 u_1 D_6] du_1 dv_1 \\ + a_n^5 \iint K_2'^2(u_1, v_1) \lambda (F_{x_0}^{-1}(\lambda))^2 D_4 du_1 dv_1 (\int u_2^2 K^*(u_2) du_2) \\ + O(a_n^6).$$

Similarly,

$$\begin{aligned}
E(l_{21}p_1) &= E\left[K^*\left(\frac{G(x_1)-G(x_0)}{a_n}\right)K_2\left(\frac{G(x_1)-G(x_0)}{a_n}, \frac{F_{x_0}(y_1)-\lambda}{a_n}\right)\right. \\
&\quad \left.K_2\left(\frac{G(x_2)-G(x_0)}{a_n}, \frac{F_{x_0}(y_2)-\lambda}{a_n}\right)Y_1Y_2I(Y_1 \leq Y_2)F_{x_0}(Y_1)\right] \\
&= \iiint K^*\left(\frac{G(x_1)-G(x_0)}{a_n}\right)K_2\left(\frac{G(x_1)-G(x_0)}{a_n}, \frac{F_{x_0}(y_1)-\lambda}{a_n}\right) \\
&\quad K_2\left(\frac{G(x_2)-G(x_0)}{a_n}, \frac{F_{x_0}(y_2)-\lambda}{a_n}\right)Y_1Y_2I(Y_1 \leq Y_2)F_{x_0}(Y_1)dH(x_1, y_1)dH(x_2, y_2) \\
&= a_n^4 \int \int \int \int K^*(u_1)K'_2(u_1, v_1)K'_2(u_2, v_2)[F_{x_0}^{-1}(\lambda) + a_n v_1 D_1 + a_n^2 v_1^2 D_2 + O(a_n^3)] \\
&\quad [F_{x_0}^{-1}(\lambda) + a_n v_2 D_1 + a_n^2 v_2^2 D_2 + O(a_n^3)][\lambda + a_n v_1] \\
&\quad [1 + a_n u_1 D_6 + a_n^2 u_1^2 D_7 + a_n^2 u_1 v_1 D_8 + O(a_n^3)] \\
&\quad [1 + a_n u_2 D_6 + a_n^2 u_2^2 D_7 + a_n^2 u_2 v_2 D_8 + O(a_n^3)] du_1 dv_1 du_2 dv_2,
\end{aligned}$$

for $n > n_0$.

So, we have

$$\begin{aligned}
(4.4.79) \quad E(l_{21}p_1) &= a_n^4 \iiint K^*(u_1)K'_2(u_1, v_1)K'_2(u_2, v_2)I(v_1 \leq v_2)\lambda(F_{x_0}^{-1}(\lambda))^2 du_1 dv_1 du_2 dv_2 \\
&\quad + a_n^5 \iiint K^*(u_1)K'_2(u_1, v_1)K'_2(u_2, v_2)I(v_1 \leq v_2)[\lambda F_{x_0}^{-1}(\lambda)(v_1 + v_2)D_1 \\
&\quad + (F_{x_0}^{-1}(\lambda))^2 v_1 + \lambda(F_{x_0}^{-1}(\lambda))^2(u_1 + u_2)D_6] du_1 dv_1 du_2 dv_2 \\
&\quad + O(a_n^6).
\end{aligned}$$

Also,

$$(4.4.80) \quad E(l_{12})E(p_1) = \iiint K^* \left(\frac{G(x_2)-G(x_0)}{a_n}, \frac{G(x_1)-G(x_0)}{a_n}, \frac{F_{x_0}(y_1)-\lambda}{a_n} \right) \\ K_2' \left(\frac{G(x_3)-G(x_0)}{a_n}, \frac{F_{x_0}(y_3)-\lambda}{a_n} \right) y_1 y_3 I(y_2 \leq y_1) \\ F_{x_0}(y_3) dH(x_1, y_1) dH(x_2, y_2) dH(x_3, y_3).$$

By transformation of variables as in (4.4.18) and applying lemma 4.1.1 we easily see that, for $n > n_0$

$$E(l_{12})E(p_3) = a_n^5 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 K^*(u_2) K_2'(u_1, v_1) K_2'(u_3, v_3) \lambda^2 (F_{x_0}^{-1}(\lambda))^2 du_1 dv_1 du_3 dv_3 du_2 \\ + O(a_n^6).$$

But the first term on the R.H.S. expression is zero since $\int K_2'(u, v) du dv = 0$.

Hence, we have

$$(4.4.81) \quad E(l_{12})E(p_1) = O(a_n^6).$$

Now, (4.4.78), (4.4.79) and (4.4.81) in (4.3.76) give

$$(4.4.82) \quad \text{Cov}(n^{1/2} a_n I_{312}, n^{1/2} a_n I_{30}) \\ = \frac{n-1}{na_n^2} \int K_2'^2(u_1, v_1) \lambda^2 (F_{x_0}^{-1}(\lambda))^2 du_1 dv_1 \\ + \frac{n-1}{na_n} \int K_2'(u_1, v_1) [2\lambda^2 F_{x_0}^{-1}(\lambda) v_1 D_1 + 2\lambda (F_{x_0}^{-1}(\lambda))^2 v_1 + \lambda^2 (F_{x_0}^{-1}(\lambda))^2 u_1 D_6] \\ du_1 dv_1$$

continued

$$\begin{aligned}
& + \frac{n-1}{n} \iint K_2'^2(u_1, v_1) [2\lambda^2 F_{x_0}^{-1}(\lambda) v_1^2 D_2 + \lambda^2 v_1^2 D_1 + \lambda^2 (F_{x_0}^{-1}(\lambda))^2 (u_1^2 D_7 + u_1 v_1 D_8) \\
& \quad + 4\lambda F_{x_0}^{-1}(\lambda) v_1^2 D_1 + 2\lambda^2 F_{x_0}^{-1}(\lambda) v_1 u_1 D_1 D_6 + (F_{x_0}^{-1}(\lambda))^2 v_1^2 \\
& \quad + 2\lambda (F_{x_0}^{-1}(\lambda))^2 v_1 u_1 D_6] du_1 dv_1 \\
& + \frac{n-1}{n} \iint K_2'(u_1, v_1) \lambda (F_{x_0}^{-1}(\lambda))^2 D_4 du_1 dv_1 (\int u_2^2 K^*(u_2) du_2) \\
& + \frac{n-1}{na_n} \iiint K^*(u_1) K_2'(u_1, v_1) K_2'(u_2, v_2) I(v_1 \leq v_2) \lambda (F_{x_0}^{-1}(\lambda))^2 du_1 dv_1 du_2 dv_2 \\
& + \frac{n-1}{n} \iiint K^*(u_1) K_2'(u_1, v_1) K_2'(u_2, v_2) I(v_1 \leq v_2) \\
& \quad [\lambda F_{x_0}^{-1}(\lambda) (v_1 + v_2) D_1 + (F_{x_0}^{-1}(\lambda))^2 v_1 + \lambda (F_{x_0}^{-1}(\lambda))^2 (u_1 + u_2) D_6] \\
& \quad du_1 dv_1 du_2 dv_2 \\
& + O(a_n).
\end{aligned}$$

From (4.4.70), (4.4.75), (4.4.82) and (4.4.46), it follows that

$$\begin{aligned}
& \text{Var}[n^{1/2} a_n (I_{312} - I_{30})] \\
& = \frac{(n-1)(n-2)}{n^2} 2(F_{x_0}^{-1}(\lambda))^2 \iiint v_1 I(v_1 \leq v_2) K_2'(u_1, v_1) K_2'(u_2, v_2) du_1 dv_1 du_2 dv_2 \\
& \quad (\int K^{*2}(u_3) du_3) \\
& + O(a_n) + O\left(\frac{1}{n}\right) + \left(\frac{1}{na_n}\right) + O\left(\frac{1}{na_n^2}\right) + O\left(\frac{1}{na_n^3}\right),
\end{aligned}$$

and by letting $n \rightarrow \infty$, we have

$$(4.4.83) \quad \lim_{n \rightarrow \infty} \text{Var}[n^{1/2} a_n (I_{312} - I_{30})]$$

$$= 2(F_{x_0}^{-1}(\lambda))^2 (\int K^{*2}(u_3) du_3) \iiint v_1 I(v_1 \leq v_2) K_2'(u_1, v_1) K_2'(u_2, v_2) du_1 dv_1 du_2 dv_2.$$

$$\begin{aligned}
\text{But, } I &= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 v_1 I(v_1 \leq v_2) K'_2(u_1, v_1) K'_2(u_2, v_2) du_1 dv_1 du_2 dv_2 \\
&= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 v_1 K'_2(u_1, v_1) \left(\int_{v_1}^1 K'_2(u_2, v_2) dv_2 \right) du_1 dv_1 du_2 \\
&= \int_{-1}^1 \int_{-1}^1 v_1 K'_2(u_1, v_1) (-K(u_2, v_1)) du_1 dv_1 du_2,
\end{aligned}$$

since $K(u, 1) = 0$ for all u , by hypothesis (see (4.1.3)).

$$\begin{aligned}
\text{Therefore, } I &= - \int_{-1}^1 \int_{-1}^1 v_1 K_2(v_1) K'_2(u_1, v_1) du_1 dv_1 \\
&= - \int_{-1}^1 \left[\int_{-1}^1 (v_1 K_2(v_1)) K'_2(u_1, v_1) dv_1 \right] du_1 \\
&= \int_{-1}^1 \int_{-1}^1 K(u_1, v_1) [v_1 K'_2(v_1) + K_2(v_1)] du_1 dv_1, \text{ where } K_2(v) = \int_{-1}^1 K(u, v) du,
\end{aligned}$$

on integration by parts and using (4.1.3). Integrating out u_1 , we get

$$\begin{aligned}
I &= \int_{-1}^1 K_2(v_1) [v_1 K'_2(v_1) + K_2(v_1)] dv_1 \\
&= \int_{-1}^1 v_1 K_2(v_1) K'_2(v_1) dv_1 + \int_{-1}^1 K_2^2(v_1) dv_1.
\end{aligned}$$

Using integration by parts to the first integral on the R.H.S. and using (4.1.3) we obtain

$$(4.4.84) \quad \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 v_1 I(v_1 \leq v_2) K'_2(u_1, v_1) K'_2(u_2, v_2) du_1 dv_1 du_2 dv_2 = \frac{1}{2} \int_{-1}^1 K_2^2(v_1) dv_1$$

and hence from (4.4.83)

$$\lim_{n \rightarrow \infty} \text{Var}[n^{1/2} a_n(I_{312} - I_{30})] = F_{x_0}^{-1}(\lambda)^2 \int \int K^2(u) K_2^2(v) du dv.$$

This completes the proof of lemma 4.4.2. $\square$

LEMMA 4.4.3. Under the hypothesis of lemma 4.4.1 $\text{Var}(n^{1/2}a_n I_1)$ converges
to $(F_{x_0}^{-1}(\lambda))^2 \iint K^2(u,v) du dv$ as $n \rightarrow \infty$ where I_1 is as given in (4.4.1).

PROOF: From (4.4.1)

$$(4.4.85) \quad \text{Var}(n^{1/2}a_n I_1) = \frac{1}{na_n^2} \text{Var}(\sum q_1) \\ = \frac{1}{a_n^2} \text{Var}(q_1),$$

where

$$(4.4.86) \quad q_1 = K\left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n}\right) y_1.$$

Therefore

$$E(q_1^2) = \iint K^2\left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n}\right) y_1^2 dH(x_1, y_1)$$

and by making the transformation of variables (4.4.13) and applying lemma 4.1.1, we can write, for $n > n_0$

$$(4.4.87) \quad E(q_1^2) = a_n^2 \int_{-1}^1 \int_{-1}^1 K^2(u_1, v_1) [F_{x_0}^{-1}(\lambda) + a_n v_1 D_1 + a_n^2 v_1^2 D_2 + O(a_n^3)]^2 \\ [1 + a_n u_1 D_6 + a_n^2 u_1^2 D_7 + a_n^2 u_1 v_1 D_8 + O(a_n^3)] du_1 dv_1 \\ = a_n^2 (F_{x_0}^{-1}(\lambda))^2 \iint K^2(u_1, v_1) du_1 dv_1 + O(a_n^3).$$

By similar computation, we easily see that

$$(4.4.88) \quad E^2(q_1) = O(a_n^4).$$

Hence, from (4.4.85), (4.4.86) and (4.4.88), we have

$$\text{Var}(n^{1/2} a_n I_1) = (F_{x_0}^{-1}(\lambda))^2 \iint K^2(u, v) du dv + O(a_n),$$

from which the lemma follows immediately by letting $n \rightarrow \infty$. $\square$

In order to find the limiting value of $\text{Var}(n^{1/2} a_n S_{n, x_0}(\lambda))$ from (4.4.2), all the required variance terms have been computed in lemmas 4.4.1 to 4.4.3.

The only covariance term that we need to compute is

$\text{Cov}(n^{1/2} a_n I_1, n^{1/2} a_n (I_{312} - I_{30}))$, since it follows from lemma 4.4.1 that all the remaining covariances go to zero as n goes to ∞ .

LEMMA 4.4.4. Under the hypothesis of lemma 4.1.1,

$\text{Cov}(n^{1/2} a_n I_1, n^{1/2} a_n (I_{312} - I_{30}))$ converges to $\sigma_{12}(x_0, \lambda)$ as $n \rightarrow \infty$ where

$$(4.4.89) \quad \sigma_{12}(x_0, \lambda) = -(F_{x_0}^{-1}(\lambda))^2 \iint K^*(u) K_2(v) K(u, v) du dv$$

and I_1, I_{312}, I_{30} are as given by (4.4.1), (4.4.4) and (4.4.1) respectively.

PROOF: We shall write

$$\begin{aligned} (4.4.90) \quad \text{Cov}(n^{1/2} a_n I_1, n^{1/2} a_n (I_{312} - I_{30})) \\ = \text{Cov}(n^{1/2} a_n I_1, n^{1/2} a_n I_{312}) - \text{Cov}(n^{1/2} a_n I_1, n^{1/2} a_n I_{30}). \end{aligned}$$

As in (4.4.76), we have

$$\begin{aligned}
 (4.4.91) \quad \text{Cov}(n^{1/2}a_n I_1, n^{1/2}a_n I_{312}) &= na_n^2 \cdot \text{Cov}\left(\frac{1}{na_n^2} \sum_m q_m, \frac{1}{n^2 a_n^4} \sum_{i \neq j} \sum_l l_{ij}\right) \\
 &= \frac{1}{n^2 a_n^4} [n(n-1) \text{Cov}(q_1, l_{12}) + n(n-1) \text{Cov}(q_1, l_{21})] \\
 &= \frac{n-1}{na_n^4} \text{Cov}(q_1, l_{12}) + \frac{n-1}{na_n^4} \text{Cov}(q_1, l_{21})
 \end{aligned}$$

where q_m 's are defined by (4.4.86) and l_{ij} 's are defined by (4.4.47).

Proceeding exactly as in (4.4.76), we have

$$\begin{aligned}
 E(l_{12}q_1) &= \iiint K^* \left(\frac{G(x_2) - G(x_0)}{a_n}, \frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) \\
 &\quad K \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1^2 F_{x_2}(y_1) dH(x_1, y_1) dG(x_2).
 \end{aligned}$$

Transformation of variables (4.4.18) together with lemma 4.1.1 yields, for $n > n_0$

$$\begin{aligned}
 (4.4.92) \quad E(l_{12}q_1) &= a_n^3 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 K^*(u_2) K'_2(u_1, v_1) K(u_1, v_1) \\
 &\quad [F_{x_0}^{-1}(\lambda) + a_n v_1 D_1 + a_n^2 v_1^2 D_2 + O(a_n^3)]^2 \\
 &\quad [\lambda + a_n u_2 D_3 + a_n v_1 + a_n^2 u_2^2 D_4 + a_n^2 u_2 v_1 D_5 + O(a_n^3)] \\
 &\quad [1 + a_n u_1 D_6 + a_n^2 u_1^2 D_7 + a_n^2 u_1 v_1 D_8 + O(a_n^3)] du_1 dv_1 du_2 \\
 &= a_n^3 \int K'_2(u_1, v_1) K(u_1, v_1) \lambda (F_{x_0}^{-1}(\lambda))^2 du_1 dv_1 \\
 &\quad + a_n^4 \int K'_2(u_1, v_1) K(u_1, v_1) [2\lambda F_{x_0}^{-1}(\lambda) v_1 D_1 + (F_{x_0}^{-1}(\lambda))^2 v_1 + \lambda (F_{x_0}^{-1}(\lambda))^2 u_1 D_6] \\
 &\quad du_1 dv_1 \\
 &\quad + O(a_n^5),
 \end{aligned}$$

where the last equality follows by expansion and using the facts $\int_{-1}^1 K^*(u) du = 1$,

$$\int_{-1}^1 u K^*(u) du = 0.$$

$$(4.4.93) \quad E(l_{21}q_1) = \iiint K^*\left(\frac{G(x_1)-G(x_0)}{a_n}\right) K_2\left(\frac{G(x_2)-G(x_0)}{a_n}, \frac{F_{x_0}(y_2)-\lambda}{a_n}\right) \\ K\left(\frac{G(x_1)-G(x_0)}{a_n}, \frac{F_{x_0}(y_1)-\lambda}{a_n}\right) y_1 y_2 I(y_1 \leq y_2) dH(x_1, y_1) dH(x_2, y_2).$$

Again, the usual transformation of variables (4.4.18) together with lemma 4.1.1 yields, for $n > n_0$,

$$(4.4.94) \quad E(l_{21}q_1) = a_n^{-4} \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 K^*(u_1) K_2'(u_2, v_2) K(u_1, v_1) [F_{x_0}^{-1}(\lambda) + a_n v_1 D_1 + a_n^2 v_1^2 D_2 \\ + O(a_n^3)] \\ [F_{x_0}^{-1}(\lambda) + a_n v_2 D_1 + a_n^2 v_2^2 D_2 + O(a_n^3)] \\ [1 + a_n u_1 D_6 + a_n^2 u_1^2 D_7 + a_n^2 u_1 v_1 D_8 + O(a_n^3)] \\ [1 + a_n u_2 D_6 + a_n^2 u_2^2 D_7 + a_n^2 u_2 v_2 D_8 + O(a_n^3)] du_1 dv_1 du_2 dv_2 \\ = a_n^4 \iiint K^*(u_1) K_2'(u_2, v_2) K(u_1, v_1) I(v_1 \leq v_2) [F_{x_0}^{-1}(\lambda)]^2 du_1 dv_1 du_2 dv_2 \\ + O(a_n^5).$$

Also,

$$(4.4.95) \quad E(l_{12})E(q_3) = \iiint K^*\left(\frac{G(x_2)-G(x_0)}{a_n}\right) K_2\left(\frac{G(x_1)-G(x_0)}{a_n}, \frac{F_{x_0}(y_1)-\lambda}{a_n}\right) \\ K\left(\frac{G(x_3)-G(x_0)}{a_n}, \frac{F_{x_0}(y_3)-\lambda}{a_n}\right) y_1 y_3 I(y_2 \leq y_1) dH(x_1, y_1) \\ dH(x_2, y_2) dH(x_3, y_3).$$

After the usual transformation of variables, we can easily see that

$$(4.4.96) \quad E(\ell_{12})E(q_3) = O(a_n^5).$$

Now, using (4.4.92) to (4.4.96) in (4.4.91), we get

$$\begin{aligned} (4.4.97) \quad \text{Cov}(n^{1/2}a_n I_{11}, n^{1/2}a_n I_{312}) &= \frac{n-1}{n} \iint K'_2(u_1, v_1) K(u_1, v_1) \lambda (F_{x_0}^{-1}(\lambda))^2 du_1 dv_1 \\ &+ \frac{n-1}{n} \iint K'_2(u_1, v_1) K(u_1, v_1) [2\lambda F_{x_0}^{-1}(\lambda) v_1 D_1 + (F_{x_0}^{-1}(\lambda))^2 v_1 \\ &\quad + \lambda (F_{x_0}^{-1}(\lambda))^2 u_1 D_6] du_1 dv_1 \\ &+ \frac{n-1}{n} \iiint K^*(u_1) K'_2(u_2, v_2) K(u_1, v_1) I(v_1 \leq v_2) (F_{x_0}^{-1}(\lambda))^2 du_1 dv_1 du_2 dv_2 \\ &+ O(a_n). \end{aligned}$$

Observe that

$$\begin{aligned} (4.4.98) \quad \text{Cov}(n^{1/2}a_n I_{11}, n^{1/2}a_n I_{30}) &= na_n^2 \text{Cov}\left(\frac{1}{na_n^2} \sum_i q_i, \frac{1}{na_n^3} \sum_j p_j\right) \\ &= \frac{1}{na_n^3} \sum_i \sum_j \text{Cov}(q_i, p_j) = \frac{1}{na_n^3} \sum_i \text{Cov}(q_i, p_i) \\ &= \frac{1}{a_n^3} \text{Cov}(q_1, p_1), \end{aligned}$$

where q_i 's are defined by (4.4.86) and p_j 's are defined by (4.4.72).

$$\begin{aligned} \text{We have } E(q_1 p_1) &= \iint K\left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n}\right) K_2\left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n}\right) \\ &\quad y_1^2 F_{x_0}(y_1) dH(x_1, y_1). \end{aligned}$$

Transformation of variables (4.4.13) together with lemma 4.1.1 gives, for $n > n_0$,

$$\begin{aligned}
 (4.4.99) \quad E(q_1 p_1) &= a_n^2 \int_{-1}^1 \int_{-1}^1 K(u_1, v_1) K'_2(u_1, v_1) [F_{x_0}^{-1}(\lambda) + a_n v_1 D_1 + a_n^2 v_1^2 D_2 + O(a_n^3)]^2 \\
 &\quad [\lambda + a_n v_1] [1 + a_n u_1 D_6 + a_n^2 u_1^2 D_7 + a_n u_1 v_1 D_8 + O(a_n^3)] du_1 dv_1 \\
 &= a_n^2 \iint K(u_1, v_1) K'_2(u_1, v_1) \lambda (F_{x_0}^{-1}(\lambda))^2 du_1 dv_1 \\
 &\quad + a_n^3 \iint K(u_1, v_1) K'_2(u_1, v_1) [2\lambda F_{x_0}^{-1}(\lambda) v_1 D_1 + (F_{x_0}^{-1}(\lambda))^2 v_1 + \lambda (F_{x_0}^{-1}(\lambda))^2 \\
 &\quad \quad \quad u_1 D_6] du_1 dv_1 \\
 &\quad + O(a_n^4).
 \end{aligned}$$

Also,

$$\begin{aligned}
 E(q_1)E(p_1) &= \iiint K_2\left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n}\right) K\left(\frac{G(x_2) - G(x_0)}{a_n}, \frac{F_{x_0}(y_2) - \lambda}{a_n}\right) \\
 &\quad y_1 y_2 F_{x_0}(y_1) dH(x_1, y_1) dH(x_2, y_2),
 \end{aligned}$$

and by the usual transformation of variables, we immediately see that

$$(4.4.100) \quad E(q_1)E(p_1) = O(a_n^4).$$

It now follows from (4.4.98), (4.4.99) and (4.4.100) that

$$\begin{aligned}
 (4.4.101) \quad \text{Cov}(n^{1/2} a_n I_1, n^{1/2} a_n I_{30}) \\
 &= \frac{1}{a_n} \iint K(u_1, v_1) K'_2(u_1, v_1) \lambda (F_{x_0}^{-1}(\lambda))^2 du_1 dv_1 \\
 &\quad + \iint K(u_1, v_1) K'_2(u_1, v_1) [2\lambda F_{x_0}^{-1}(\lambda) v_1 D_1 + (F_{x_0}^{-1}(\lambda))^2 v_1 + \lambda (F_{x_0}^{-1}(\lambda))^2 \\
 &\quad \quad \quad u_1 D_6] du_1 dv_1 \\
 &\quad + O(a_n).
 \end{aligned}$$

(4.4.97) and (4.4.101) in (4.4.90) yield

$$\begin{aligned} & \text{Cov}(n^{1/2} a_n I_1, n^{1/2} a_n (I_{312} - I_{30})) \\ &= \frac{n-1}{n} \iiint K^*(u_1) K'_2(u_2, v_2) K(u_1, v_1) I(v_1 \leq v_2) (F_{x_0}^{-1}(\lambda))^2 du_1 dv_1 du_2 dv_2 \\ &+ o(a_n) + o\left(\frac{1}{n}\right) + o\left(\frac{1}{na_n}\right), \end{aligned}$$

and by letting $n \rightarrow \infty$, we have

$$\begin{aligned} (4.4.102) \quad & \lim_{n \rightarrow \infty} \text{Cov}(n^{1/2} a_n I_1, n^{1/2} a_n (I_{312} - I_{30})) \\ &= (F_{x_0}^{-1}(\lambda))^2 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 K^*(u_1) K'_2(u_2, v_2) K(u_1, v_1) I(v_1 \leq v_2) \\ &\quad du_1 dv_1 du_2 dv_2. \end{aligned}$$

But as in (4.4.83) we can write

$$\begin{aligned} (4.4.103) \quad & \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 K^*(u_1) K'_2(u_2, v_2) K(u_1, v_1) I(v_1 \leq v_2) du_1 dv_1 du_2 dv_2 \\ &= \int_{-1}^1 \int_{-1}^1 K^*(u_1) K(u_1, v_1) \left(\int_{v_1}^1 K'_2(u_2, v_2) dv_2 \right) du_1 dv_1 du_2 \\ &= - \int_{-1}^1 \int_{-1}^1 K^*(u_1) K(u_1, v_1) K(u_2, v_1) du_1 dv_1 dv_2, \\ &\quad \text{since } K(u, 1) = 0 \text{ for all } u, \\ &= - \int_{-1}^1 \int_{-1}^1 K^*(u_1) K(u_1, v_1) K_2(v_1) du_1 dv_1, \end{aligned}$$

by integrating out u_2 .

Hence we have (4.4.89) and the proof of the lemma 4.4.4 is complete. $\square$

Now, from equation (4.4.2) and lemmas 4.4.1 to 4.4.4, it follows that

$$\begin{aligned}
(4.4.104) \quad \lim_{n \rightarrow \infty} \text{Var}(n^{1/2} a_n S_{n, x_0}(\lambda)) &= \lim_{n \rightarrow \infty} [\text{Var}(n^{1/2} a_n I_1) + \text{Var}(n^{1/2} a_n (I_{312} - I_{30})) \\
&\quad + 2\text{Cov}(n^{1/2} a_n I_1, n^{1/2} a_n (I_{312} - I_{30}))] \\
&= (F_{x_0}^{-1}(\lambda))^2 \iint K^2(u, v) du dv + (F_{x_0}^{-1}(\lambda))^2 \iint K^{*2}(u) K_2^2(v) du dv \\
&\quad - 2(F_{x_0}^{-1}(\lambda))^2 \iint K^*(u) K_2(v) K(u, v) du dv \\
&= (F_{x_0}^{-1}(\lambda))^2 \iint [K(u, v) - K^*(u) K_2(v)]^2 du dv.
\end{aligned}$$

4.5. CONSISTENCY OF $Q_{n, x_0}(\lambda)$. We now give the proof of Theorem 4.1.1 in this section which proves the consistency of $Q_{n, x_0}(\lambda)$.

PROOF OF THEOREM 4.1.1. From (4.1.19) we can write

$$(4.5.1) \quad Q_{n, x_0}(\lambda) - Q_{x_0}(\lambda) = [S_{n, x_0}(\lambda) - Q_{x_0}(\lambda)] + \sum_{m=2}^3 \zeta_{nm} + \sum_{m=2}^4 \eta_{nm}.$$

Because of lemma 4.3.1 it remains to show that $[S_{n, x_0}(\lambda) - Q_{x_0}(\lambda)]$ converges to zero in probability as $n \rightarrow \infty$. Relation (4.4.104) implies that

$\text{Var}[S_{n, x_0}(\lambda) - Q_{x_0}(\lambda)]$ goes to zero as $n \rightarrow \infty$. Hence it suffices to show that

$E[S_{n, x_0}(\lambda) - Q_{x_0}(\lambda)]$ goes to zero as $n \rightarrow \infty$. From (4.4.2) we have

$$S_{n, x_0}(\lambda) = I_1 + I_2 + I_{311} + I_{312} + I_{321} + I_{322} - I_{30}$$

so that we can write

$$\begin{aligned}
(4.5.2) \quad E[S_{n, x_0}(\lambda) - Q_{x_0}(\lambda)] &= E(I_1 - Q_{x_0}(\lambda)) + E(I_2) + E(I_{311}) \\
&\quad + E(I_{312} - I_{30}) + E(I_{321}) + E(I_{322}).
\end{aligned}$$

We shall now show that each of the expectations on the R.H.S. of (4.5.2) goes to zero as $n \rightarrow \infty$.

We have

$$(4.5.3) \quad E(I_2) = \frac{1}{na_n^3} \sum E(t_1)$$

where t_1 's are as given by (4.4.10). We also have, from (4.4.11),

$$\begin{aligned} E(t_1) &= \frac{1}{n} \iint [I(x_1 > x_0) - (G(x_1) - G(x_0))] K_1' \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 dH(x_1, y_1) \\ &= O\left(\frac{1}{n}\right) \end{aligned}$$

since K_1' is bounded and $E|Y| < \infty$. So it follows from (4.5.3) that

$$(4.5.4) \quad E(I_2) = O\left(\frac{1}{na_n^3}\right)$$

which goes to zero as $n \rightarrow \infty$ since $na_n^3 \rightarrow \infty$. From (4.4.3) we have

$$(4.5.5) \quad E(I_{311}) = \frac{1}{n^2 a_n^4} \sum E(l_1)$$

$$\text{where } l_1 = K^* \left(\frac{G(X_1) - G(x_0)}{a_n} \right) K_2' \left(\frac{G(X_1) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_1) - \lambda}{a_n} \right) Y_1.$$

Since K^*, K_2' are bounded and $E|Y| < \infty$ it follows from (4.5.5) that

$$(4.5.6) \quad E(I_{311}) = O\left(\frac{1}{na_n^4}\right)$$

which goes to zero as $n \rightarrow \infty$ because $na_n^4 \rightarrow \infty$.

From (4.4.4) we have

$$(4.5.7) \quad E(I_{312}) = \frac{1}{n^2 a_n^4} \sum_{i \neq j} E(l_{ij}) = \frac{n-1}{n a_n^4} E(l_{12})$$

where

$$l_{ij} = K^* \left(\frac{G(X_j) - G(x_0)}{a_n} \right) K_2' \left(\frac{G(X_i) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_1) - \lambda}{a_n} \right) Y_1 I(Y_j \leq Y_1).$$

Hence

$$\begin{aligned} E(l_{12}) &= \iiint K^* \left(\frac{G(x_2) - G(x_0)}{a_n} \right) K_2' \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 I(y_2 \leq y_1) \\ &\quad dH(x_1, y_1) dH(x_2, y_2) \\ &= \iiint K^* \left(\frac{G(x_2) - G(x_0)}{a_n} \right) K_2' \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 F_{x_2}(y_1) \\ &\quad dH(x_1, y_1) dG(x_2). \end{aligned}$$

Now, by transformation of variables (4.4.18) in the last integral and using Taylor expansions from lemma 4.1.1, we have for $n > n_0$

$$\begin{aligned} (4.5.8) \quad E(l_{12}) &= a_n^3 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 K^*(u_2) K_2'(u_1, v_1) [F_{x_0}^{-1}(\lambda) + a_n v_1 D_1 + a_n^2 v_1^2 D_2 + O(a_n^3)] \\ &\quad [\lambda + a_n u_2 D_3 + a_n v_1 + a_n^2 u_2^2 D_4 + a_n^2 u_2 v_1 D_5 + O(a_n^3)] \\ &\quad [1 + a_n u_1 D_6 + a_n^2 u_1^2 D_7 + a_n^2 u_1 v_1 D_8 + O(a_n^3)] du_1 dv_1 du_2 \\ &= a_n^3 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 K^*(u_2) K_2'(u_1, v_1) \lambda F_{x_0}^{-1}(\lambda) du_1 dv_1 du_2 \\ &\quad + a_n^4 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 K^*(u_2) K_2'(u_1, v_1) [\lambda v_1 D_1 + F_{x_0}^{-1}(\lambda) v_1 + \lambda F_{x_0}^{-1}(\lambda) u_1 D_6] du_1 dv_1 du_2 \\ &\quad + O(a_n^5). \end{aligned}$$

Since $\int_{-1}^1 K_2'(u,v) du dv = 0$, the first integral in the last expression vanishes and from (4.5.7) and (4.5.8) we have

$$(4.5.9) \quad E(I_{312}) = \frac{n-1}{n} \int_{-1}^1 \int_{-1}^1 K_2'(u_1, v_1) [\lambda v_1 D_1 + F_{x_0}^{-1}(\lambda) v_1 + \lambda F_{x_0}^{-1}(\lambda) u_1 D_6] du_1 dv_1 + O(a_n).$$

Now from (4.1.1) we write

$$(4.5.10) \quad E(I_{30}) = \frac{1}{na_n^3} [E(p_1)] = \frac{1}{a_n^3} E(p_1)$$

where

$$p_1 = K_2' \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 F_{x_0}(y_1).$$

Therefore

$$E(p_1) = \int \int K_2' \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 F_{x_0}(y_1) dH(x_1, y_1).$$

By making transformation of variables (4.4.13) and using Taylor expansions from lemma 4.1.1, we see that, for $n > n_0$

$$(4.5.11) \quad E(p_1) = a_n^2 \int_{-1}^1 \int_{-1}^1 K_2'(u_1, v_1) [F_{x_0}^{-1}(\lambda) + a_n v_1 D_1 + a_n^2 v_1^2 D_2 + O(a_n^3)] [\lambda + a_n v_1] [1 + a_n u_1 D_6 + a_n^2 u_1^2 D_7 + a_n^2 u_1 v_1 D_8 + O(a_n^3)] du_1 dv_1 =$$

$$= a_n^2 \int_{-1}^1 \int_{-1}^1 K_2'(u_1, v_1) \lambda F_{x_0}^{-1}(\lambda) du_1 dv_1$$

$$+ a_n^3 \int_{-1}^1 \int_{-1}^1 K_2'(u_1, v_1) [\lambda v_1 D_1 + F_{x_0}^{-1}(\lambda) v_1 + \lambda F_{x_0}^{-1}(\lambda) u_1 D_6] du_1 dv_1$$

$$+ O(a_n^4).$$

As in (4.5.8), the first integral in the last expression vanishes and we have from (4.5.10) and (4.5.11)

$$(4.5.12) \quad E(I_{30}) = \int_{-1}^1 \int_{-1}^1 K_2'(u_1, v_1) [\lambda v_1 D_1 + F_{x_0}^{-1}(\lambda) v_1 + \lambda F_{x_0}^{-1}(\lambda) u_1 D_6] du_1 dv_1 \\ + O(a_n).$$

From (4.5.9) and (4.5.12) it now follows that

$$(4.5.13) \quad E(I_{312} - I_{30}) = O\left(\frac{1}{n}\right) + O(a_n)$$

which goes to zero as $n \rightarrow \infty$.

From (4.4.5) we have

$$(4.5.14) \quad E(I_{321}) = \frac{1}{n^2 a_n^5} \sum E(t_{ii})$$

$$\text{where } t_{ii} = [G_n(X_i) - G_n(x_0) - G(X_i) + G(x_0)] K^* \left(\frac{G(X_i) - G(x_0)}{a_n} \right)$$

$$K_2' \left(\frac{G(X_i) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_i) - \lambda}{a_n} \right) Y_i.$$

Hence for $i=1, 2, \dots, n$

$$(4.5.15) \quad E(t_{ii}) = \int \int \frac{1}{n} [I(x_1 > x_0) - G(x_1) - G(x_0)] K^* \left(\frac{G(x_1) - G(x_0)}{a_n} \right)$$

$$K_2' \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 dH(x_1, y_1)$$

$$= O\left(\frac{1}{n}\right),$$

since K^* , K_2' are bounded and $E|Y| < \infty$. From (4.5.14) and (4.5.15) we now have

$$(4.5.16) \quad E(I_{321}) = O\left(\frac{1}{n^2 a_n^5}\right)$$

which goes to zero as $n \rightarrow \infty$. From (4.4.6) we write

$$(4.5.17) \quad E(I_{322}) = \frac{1}{n^2 a_n^5} \sum_{i \neq j} E(t_{ij}),$$

where t_{ij} 's are given by (4.4.32) and from (4.4.33) we have

$E(t_{ij}) = O\left(\frac{a_n^3}{n}\right)$ for $i, j = 1, \dots, n, i \neq j$. Hence it follows from (4.5.17) that

$$(4.5.18) \quad E(I_{322}) = O\left(\frac{1}{n a_n^2}\right)$$

which goes to zero as $n \rightarrow \infty$.

Now it remains to show that $E(I_1 - Q_{x_0}(\lambda)) \rightarrow 0$ as $n \rightarrow \infty$. From (4.4.1) we have

$$I_1 = \frac{1}{n a_n^2} \sum K\left(\frac{G(x_i) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_i) - \lambda}{a_n}\right) Y_i$$

and hence

$$E(I_1) - Q_{x_0}(\lambda) = \frac{1}{a_n^2} \iint K\left(\frac{G(x) - G(x_0)}{a_n}, \frac{F_{x_0}(y) - \lambda}{a_n}\right) y dH(x, y) - Q_{x_0}(\lambda).$$

By transformation of variables (4.4.13) and using Taylor expansions from lemma 4.1.1, we get, for $n > n_0$

$$\begin{aligned}
 (4.5.19) \quad E(I_1) - Q_{x_0}(\lambda) &= \int_{-1}^1 \int_{-1}^1 K(u,v) [F_{x_0}^{-1}(\lambda) + a_n v D_1 + a_n^2 v^2 D_2 + O(a_n^3)] \\
 &\quad [1 + a_n u D_6 + a_n^2 u^2 D_7 + a_n^2 u v D_8 + O(a_n^3)] du_1 dv_1 \\
 &\quad - Q_{x_0}(\lambda).
 \end{aligned}$$

Since $\iint K(u,v) du dv = 1$, $\iint u K(u,v) du dv = 0 = \iint v K(u,v) du dv$ and (by assumptions (4.1.3) to (4.1.5)) and $F_{x_0}^{-1}(\lambda) \equiv Q_{x_0}(\lambda)$ it follows that

$$(4.5.20) \quad E(I_1) - Q_{x_0}(\lambda) = O(a_n^2)$$

which goes to zero as $n \rightarrow \infty$.

Now (4.5.4), (4.5.6), (4.5.13), (4.5.16), (4.5.18) and (4.5.20) together implies that $E[S_{n,x_0}(\lambda) - Q_{x_0}(\lambda)] \rightarrow 0$ as $n \rightarrow \infty$. This completes the proof of Theorem 4.1.1. $\square$

CHAPTER V

ASYMPTOTIC NORMALITY OF $Q_{n,x_0}(\lambda)$

In this chapter we prove the asymptotic normality of the conditional quantile estimator $Q_{n,x_0}(\lambda)$, in a suitably normalized form where

$Q_{n,x_0}(\lambda)$ is defined by (4.1.1). We continue to use the representation (4.1.9) of $Q_{n,x_0}(\lambda)$, namely,

$$Q_{n,x_0}(\lambda) = S_{n,x_0}(\lambda) + R_n + R'_n$$

where $S_{n,x_0}(\lambda)$ is as given by (4.1.21) but the higher order terms R_n and R'_n are comprised of more terms now, than before. First we show that the higher order terms (after proper normalization of $Q_{n,x_0}(\lambda)$) converge to zero in probability, as n goes to ∞ , so that it then suffices to prove the asymptotic normality of $S_{n,x_0}(\lambda)$ (in the normalized form). Our method of proof of this is essentially the same as that used in the case of the statistic $\frac{1}{n} \sum_{i=1}^n J\left(\frac{1}{n}, \frac{R_{in}}{n}\right) Y_{[i:n]}$ in Chapters 2 and 3. We shall use Hajek's projection lemma (lemma 2.1.1) to approximate $S_{n,x_0}(\lambda)$ in mean square by a sum of iid rv's and the asymptotic normality of the latter is then proved along classical lines. In section 5.1, the main result of this chapter, namely, the asymptotic normality of $Q_{n,x_0}(\lambda)$ is stated as Theorem 5.1.1. and Corollary 5.1.1. In section 5.2. we establish the asymptotic equivalence in mean square of $S_{n,x_0}(\lambda)$ to that of a sum of iid rv's, while using some of the results obtained in chapter 4. The proofs of the Theorem and the Corollary are given in section 5.3.

5.1. PROJECTION $\hat{S}_{n,x_0}(\lambda)$ OF $S_{n,x_0}(\lambda)$. In this section we state our main results of this chapter as Theorem 5.1.1 and Corollary 5.1.1 and obtain the projection of $S_{n,x_0}(\lambda)$.

THEOREM 5.1.1. Let $Q_{n,x_0}(\lambda)$, $0 < \lambda < 1$ be as defined by (4.1.1) with K and K^* satisfying assumptions (4.1.3) to (4.1.6) with $v = 6$ and $v^* = 4$. Suppose the joint distribution function H satisfies assumptions (4.1.11) to (4.1.15). Then for each sequence $\{a_n\}$ of positive numbers with $a_n \rightarrow 0$, $na_n^4 \rightarrow \infty$ but $na_n^5 \rightarrow 0$ we have

$$(5.1.1) \quad n^{1/2} a_n [Q_{n,x_0}(\lambda) - \bar{Q}_{n,x_0}] \rightarrow N(0, \sigma_0^2(x_0, \lambda)) \text{ in distribution as } n \rightarrow \infty, \text{ provided } \sigma_0^2(x_0, \lambda) > 0, \text{ where}$$

$$(5.1.2) \quad \bar{Q}_{n,x_0}(\lambda) = a_n^{-2} \iint K\left(\frac{G(x) - G(x_0)}{a_n}, \frac{F_{x_0}(y) - \lambda}{a_n}\right) y dH(x, y)$$

$$(5.1.3) \quad \sigma_0^2(x_0, \lambda) = [F_{x_0}^{-1}(\lambda)]^2 \iint [K(u, v) - K^*(v) K_2(v)]^2 du dv \quad \text{and}$$

$$(5.1.3 a) \quad K_2(v) = \int K(u, v) du.$$

COROLLARY 5.1.1 Under the hypothesis of Theorem 5.1.1 we have

$$(5.1.4) \quad n^{1/2} a_n [Q_{n,x_0}(\lambda) - Q_{x_0}(\lambda)] \rightarrow N(0, \sigma_0^2(x_0, \lambda)) \text{ in distribution as } n \rightarrow \infty.$$

The proofs of Theorem 5.1.1 and Corollary 5.1.1 are given in section 5.3.

First observe that since K is six times differentiable and K^* is four times differentiable (under the hypothesis of Theorem 5.1.1) the following representation of $Q_{n,x_0}(\lambda)$ can be obtained exactly on the same lines as (4.1.20) was obtained:

$$(5.1.5) \quad Q_{n,x_0}(\lambda) = S_{n,x_0}(\lambda) + \sum_{m=2}^4 \tau_{nm} + \sum_{m=2}^6 \eta_{nm}$$

where $S_{n,x_0}(\lambda)$ is as given by (4.1.21), η_{nm} ($m=2,3,4$ and 5) is given by (4.1.24) and

$$(5.1.6) \quad \eta_{n6} = \frac{a_n^{-8}}{6!} \iint [G_n(x_1) - G_n(x_0) - G(x_1) + G(x_0)] \frac{\partial}{\partial t_1} + [F_{n,x_0}(y_1) - F_{x_0}(y_1)] \frac{\partial}{\partial t_2} \Big]^6 \\ K(\Delta_{x_1,n}, \Delta_{y_1,n}) y_1 dH_n(x_1, y_1),$$

$$(5.1.7) \quad \tau_{nm} = \frac{a_n^{-4-m}}{m!} \iiint K^{*(m)} \left(\frac{G(x_2) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 I(y_2 \leq y_1) \\ [G_n(x_2) - G(x_0) - G(x_2) + G(x_0)]^m dH_n(x_1, y_1) dH_n(x_2, y_2)$$

for $m = 2$ and 3 ,

$$(5.1.8) \quad \tau_{n4} = \frac{a_n^{-8}}{4!} \iiint K^{*(4)}(\Delta_{x_2,n}) K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 I(y_2 \leq y_1) \\ [G_n(x_2) - G_n(x_0) - G(x_2) + G(x_0)]^4 dH_n(x_1, y_1) dH_n(x_2, y_2).$$

The following lemma shows that the higher order terms τ_{nm} 's and η_{nm} 's in the normalized form of $Q_{n,x_0}(\lambda)$ converge to zero in probability as $n \rightarrow \infty$.

LEMMA 5.1.1. (a) Suppose assumptions (4.1.3) to (4.1.6) on K and K^* are satisfied with $v = 6$, $v^* = 4$ and assumptions (4.1.11) to (4.1.13) on the joint distribution H are satisfied. Let $\{a_n\}$ be a sequence of positive numbers with $a_n \rightarrow 0$ and $na_n^4 \rightarrow \infty$. We then have

$n^{1/2} a_n \tau_{nm} \rightarrow 0$ in probability, as $n \rightarrow \infty$ for $m = 2, 3$ and 4
where τ_{nm} are given by (5.1.7) to (5.1.8).

(b) Suppose assumptions (4.1.3) to (4.1.6) on K and K^* are satisfied with $v = 6$, $v^* = 4$ and assumptions (4.1.11) to (4.1.15) on the joint distribution H are satisfied. Let $\{a_n\}$ be a sequence of positive numbers such that $a_n \rightarrow 0$, $na_n^4 \rightarrow \infty$ but $na_n^5 \rightarrow 0$. We then have

$n^{1/2} a_n \eta_{nm} \rightarrow 0$ in probability, as $n \rightarrow \infty$ for $m = 2, \dots, 6$
where η_{nm} are given by (4.1.24) for $m = 2, 3, 4$ and 5 and η_{n6} is given by (5.1.6).

PROOF: The proof of this lemma runs parallel to that of lemma 4.3.1 and hence is omitted. $\square$

LEMMA 5.1.2. Let $S_{n,x_0}(\lambda)$ be as defined by (4.4.1). Suppose assumptions (4.1.3) to (4.1.4) on K and K^* are satisfied with $v = 1$ and $v^* = 1$.

Then the projection $\hat{S}_{n,x_0}(\lambda)$ of $S_{n,x_0}(\lambda)$, as defined by (2.1.1) is given by

$$\begin{aligned}
& + \int \frac{n-1}{n^3 a_n^5} \iint K^* \left(\frac{G(x)-G(x_0)}{a_n} \right) K_2 \left(\frac{G(x)-G(x_0)}{a_n}, \frac{F_{x_0}(y)-\lambda}{a_n} \right) \\
& \quad [I(X_1 \leq x) - I(X_1 \leq x_0) - (G(x) - G(x_0))] y dH(x, y) \\
& + \int \frac{(n-1)(n-2)}{n^3 a_n^5} \iiint K^* \left(\frac{G(x_2)-G(x_0)}{a_n} \right) K_2 \left(\frac{G(x_1)-G(x_0)}{a_n}, \frac{F_{x_0}(y_1)-\lambda}{a_n} \right) \\
& \quad [I(X_1 \leq x_2) - I(X_1 \leq x_0) - (G(x_2) - G(x_0))] y_1 I(y_2 \leq y_1) \\
& \quad dH(x_1, y_1) dH(x_2, y_2) \\
& - \int \frac{1}{n a_n^3} K_2 \left(\frac{G(X_1)-G(x_0)}{a_n}, \frac{F_{x_0}(Y_1)-\lambda}{a_n} \right) Y_1 F_{x_0}(Y_1) + C \\
& = S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + S_7 + S_8 + S_9 + S_{10} + S_{11} - S_{12} + C,
\end{aligned}$$

where C is a constant depending on K^* , K , x_0 , λ and n .

PROOF: Since $S_{n, x_0}(\lambda) = I_1 + I_2 + I_{31} + I_{32} - I_{30}$ (see 4.4.1), by the linearity property of the projection operator, we can write

$$(5.1.10) \quad \hat{S}_{n, x_0}(\lambda) = \hat{I}_1 + \hat{I}_2 + \hat{I}_{31} + \hat{I}_{32} - \hat{I}_{30}$$

where $\hat{I}_j$ denotes the projection (as defined by 2.1.1.) of I_j .

First, observe that both I_1 and I_{30} being sums of iid rv's, we have

$$(5.1.11) \quad \hat{I}_1 = I_1, \text{ and } \hat{I}_{30} = I_{30}.$$

From (4.4.1), we have

$$\begin{aligned} I_2 &= \frac{1}{na_n^3} \sum [G_n(x_1) - G_n(x_0) - G(x_1) + G(x_0)] K_1' \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 \\ &= \frac{1}{na_n^3} \sum t_1. \end{aligned}$$

By (2.1.1),

$$(5.1.12) \quad \hat{I}_2 = \sum_j E(I_2 | X_j, Y_j) - (n-1)E(I_2).$$

We now write

$$\begin{aligned} (5.1.13) \quad E(I_2 | X_n, Y_n) &= (na_n^3)^{-1} E(t_n + \sum_{i \neq n} t_i | X_n, Y_n) \\ &= (na_n^3)^{-1} E(t_n | X_n, Y_n) + (n-1)(na_n^3)^{-1} E(t_1 | X_n, Y_n), \end{aligned}$$

where

$$\begin{aligned} (5.1.14) \quad E(t_n | X_n = x_n, Y_n = y_n) &= K_1' \left(\frac{G(x_n) - G(x_0)}{a_n}, \frac{F_{x_0}(y_n) - \lambda}{a_n} \right) y_n \\ &\quad E[G_n(x_n) - G_n(x_0) - G(x_n) + G(x_0) | X_n = x_n] \\ &= K_1' \left(\frac{G(x_n) - G(x_0)}{a_n}, \frac{F_{x_0}(y_n) - \lambda}{a_n} \right) y_n \\ &\quad \frac{1}{n} [I(x_n > x_0) - (G(x_n) - G(x_0))] \end{aligned}$$

by (4.4.7), and

$$\begin{aligned}
(5.1.15) \quad E(t_1 | X_n = x_n, Y_n = y_n) &= E[G_n(X_1) - G_n(x_0) - G(X_1) + G(x_0)] \\
&= K_1' \left(\frac{G(X_1) - G(x_0)}{a_n}, \frac{F_{X_0}(Y_1) - \lambda}{a_n} \right) Y_1 | X_n = x_n, Y_n = y_n \} \\
&= \iint K_1' \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{X_0}(y_1) - \lambda}{a_n} \right) y_1 E[G_n(x_1) - G_n(x_0) \\
&\quad - G(x_1) + G(x_0)] | X_n = x_n, X_1 = x_1 \} dH(x_1, y_1) \\
&= \iint K_1' \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{X_0}(y_1) - \lambda}{a_n} \right) \frac{1}{n} [I(x_1 > x_0) + I(x_n < x_1) \\
&\quad - I(x_n < x_0) - 2(G(x_1) - G(x_0))] y_1 dH(x_1, y_1)
\end{aligned}$$

by (4.4.25).

Hence, from (5.1.13), (5.1.14) and (5.1.15), we have

$$\begin{aligned}
(5.1.16) \quad E(I_2 | X_n, Y_n) &= (n^2 a_n^3)^{-1} K_1' \left(\frac{G(X_n) - G(x_0)}{a_n}, \frac{F_{X_0}(Y_n) - \lambda}{a_n} \right) Y_n [I(X_n > x_0) - (G(X_n) - G(x_0))] \\
&\quad + (n-1)(n^2 a_n^3)^{-1} \iint K_1' \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{X_0}(y_1) - \lambda}{a_n} \right) [I(x_1 > x_0) + I(x_n < x_1) \\
&\quad - I(x_n < x_0) - 2(G(x_1) - G(x_0))] y_1 dH(x_1, y_1).
\end{aligned}$$

A symmetric argument gives the same expression for $E(I_2 | X_j, Y_j)$,

$j = 1, 2, \dots, n-1$, as on the R.H.S. of (5.1.16) with (X_n, Y_n) replaced by (X_j, Y_j) . Hence it follows (5.1.16) that

$$\begin{aligned}
 (5.1.17) \quad \hat{I}_2 &= (n^2 a_n^3)^{-1} \sum K_1' \left(\frac{G(X_1) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_1) - \lambda}{a_n} \right) Y_1 [I(X_1 > x_0) - (G(X_1) - G(x_0))] \\
 &\quad + (n-1)(n^2 a_n^3)^{-1} \iint K_1' \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) [I(X_1 < x_1) - I(X_1 < x_0) \\
 &\quad \quad - (G(x_1) - G(x_0))] y_1 dH(x_1, y_1) \\
 &\quad + C_2
 \end{aligned}$$

where C_2 is a constant (i.e. non-random) into which we have included a part of the constant term from the R.H.S. of (5.1.16) for convenience, viz.

$$\begin{aligned}
 C_2 &= \frac{n(n-1)}{n^2 a_n^3} \iint K_1' \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) [I(x_1 > x_0) - (G(x_1) - G(x_0))] \\
 &\quad \quad y_1 dH(x_1, y_1) \\
 &\quad - (n-1)E(I_2).
 \end{aligned}$$

We now proceed on the same lines as above to find $\hat{I}_{31}$ and $\hat{I}_{32}$.

By (2.1.1) we have

$$(5.1.18) \quad \hat{I}_{31} = \sum_m E(I_{31} | X_m, Y_m) - (n-1)E(I_{31})$$

where

$$\begin{aligned}
 I_{31} &= (n^2 a_n^4)^{-1} \sum_{ij} K^* \left(\frac{G(X_j) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(X_i) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_i) - \lambda}{a_n} \right) Y_i I(Y_j < Y_i) \\
 &= (n^2 a_n^4)^{-1} \sum_{ij} \ell_{ij}.
 \end{aligned}$$

We shall write

$$\begin{aligned}
 (5.1.19) \quad E(I_{31}|X_n, Y_n) &= (n^2 a_n^4)^{-1} E[\ell_{nn} + \sum_{i \neq n} \ell_{in} + \sum_{j \neq n} \ell_{nj} + \sum_{i \neq n} \ell_{ii} + \sum_{\substack{i, j \neq n \\ i \neq j}} \ell_{ij} | X_n, Y_n] \\
 &= (n^2 a_n^4)^{-1} E(\ell_{nn} | X_n, Y_n) + (n-1)(n^2 a_n^4)^{-1} E(\ell_{1n} | X_n, Y_n) \\
 &\quad + (n-1)(n^2 a_n^4)^{-1} E(\ell_{n1} | X_n, Y_n) + (n-1)(n^2 a_n^4)^{-1} E(\ell_{11} | X_n, Y_n) \\
 &\quad + (n-1)(n-2)(n^2 a_n^4)^{-1} E(\ell_{12} | X_n, Y_n).
 \end{aligned}$$

It is easy to see that

$$\begin{aligned}
 E(\ell_{nn} | X_n, Y_n) &= K^* \left(\frac{G(X_n) - G(x_0)}{a_n} \right) K_2' \left(\frac{G(X_n) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_n) - \lambda}{a_n} \right) Y_n, \\
 E(\ell_{1n} | X_n, Y_n) &= K^* \left(\frac{G(X_n) - G(x_0)}{a_n} \right) \iint K_2' \left(\frac{G(X_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) \\
 &\quad y_1 I(Y_n \leq y_1) dH(x_1, y_1), \\
 (5.1.20) \quad E(\ell_{n1} | X_n, Y_n) &= K_2 \left(\frac{G(X_n) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_n) - \lambda}{a_n} \right) Y_n \iint K^* \left(\frac{G(x_1) - G(x_0)}{a_n} \right) \\
 &\quad I(y_1 \leq Y_n) dH(x_1, y_1) \\
 E(\ell_{11} | X_n, Y_n) &= \iint K^* \left(\frac{G(x_1) - G(x_0)}{a_n} \right) K_2' \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) \\
 &\quad y_1 dH(x_1, y_1)
 \end{aligned}$$

and

$$E(l_{12}|X_n, Y_n) = \iiint K^*\left(\frac{G(x_2)-G(x_0)}{a_n}\right) K_2\left(\frac{G(x_1)-G(x_0)}{a_n}, \frac{F_{x_0}(y_1)-\lambda}{a_n}\right) \\ y_1 I(y_2 \leq y_1) dH(x_1, y_1) dH(x_2, y_2).$$

Using (5.1.20) in (5.1.19) we obtain $E(I_{31}|X_n, Y_n)$. We also observe that

$E(I_{31}|X_n, Y_n)$, $m = 1, 2, \dots, n-1$, can be obtained by replacing (X_n, Y_n) with (X_m, Y_m) in the expression for $E(I_{31}|X_n, Y_n)$. It then follows from (5.1.18) that

$$(5.1.21) \quad \hat{I}_{31} = (n^2 a_n^4)^{-1} \sum K^*\left(\frac{G(X_i)-G(x_0)}{a_n}\right) K_2\left(\frac{G(X_i)-G(x_0)}{a_n}, \frac{F_{x_0}(Y_i)-\lambda}{a_n}\right) Y_i \\ + (n-1)(n^2 a_n^4)^{-1} \sum K^*\left(\frac{G(X_i)-G(x_0)}{a_n}\right) \iint K_2\left(\frac{G(x_1)-G(x_0)}{a_n}, \frac{F_{x_0}(y_1)-\lambda}{a_n}\right) \\ y_1 I(Y_i \leq y_1) dH(x_1, y_1) \\ + (n-1)(n^2 a_n^4)^{-1} \sum K_2\left(\frac{G(X_i)-G(x_0)}{a_n}, \frac{F_{x_0}(Y_i)-\lambda}{a_n}\right) Y_i \iint K^*\left(\frac{G(x_1)-G(x_0)}{a_n}\right) \\ I(Y_i \geq y_1) dH(x_1, y_1) \\ + C_3,$$

where C_3 is the constant term (non-random).

Again by (2.1.1) we have

$$(5.1.22) \quad \hat{I}_{32} = \sum_m E(I_{32} | X_m, Y_m) - (n-1)E(I_{32})$$

where

$$I_{32} = (n^2 a_n^5)^{-1} \sum_{1j} [G_n(X_j) - G_n(x_0) - G(X_j) + G(x_0)] K^* \left(\frac{G(X_j) - G(x_0)}{a_n} \right) \\ K_2 \left(\frac{G(X_1) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_1) - \lambda}{a_n} \right) Y_1 I(Y_j < Y_1) \\ = (n^2 a_n^5)^{-1} \sum_{1j} t_{1j}.$$

As in (5.1.19) we can write

$$(5.1.23) \quad E(I_{32} | X_n, Y_n) = (n^2 a_n^5)^{-1} E(t_{nn} | X_n, Y_n) + (n-1)(n^2 a_n^5)^{-1} E(t_{1n} | X_n, Y_n) \\ + (n-1)(n^2 a_n^5)^{-1} E(t_{n1} | X_n, Y_n) + (n-1)(n^2 a_n^5)^{-1} E(t_{11} | X_n, Y_n) \\ + (n-1)(n-2)(n^2 a_n^5)^{-1} E(t_{12} | X_n, Y_n).$$

We see that

$$E(t_{nn} | X_n = x_n, Y_n = y_n) = K^* \left(\frac{G(x_n) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(x_n) - G(x_0)}{a_n}, \frac{F_{x_0}(y_n) - \lambda}{a_n} \right) y_n \\ E[G_n(x_n) - G_n(x_0) - G(x_n) + G(x_0) | X_n = x_n] \\ = K^* \left(\frac{G(x_n) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(x_n) - G(x_0)}{a_n}, \frac{F_{x_0}(y_n) - \lambda}{a_n} \right) \\ y_n \frac{1}{n} [I(x_n > x_0) - G(x_n) + G(x_0)], \quad \text{by (4.4.7),}$$

$$\begin{aligned}
& E(t_{1n} | X_n = x_n, Y_n = y_n) \\
&= K^* \left(\frac{G(x_n) - G(x_0)}{a_n} \right) \iint K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 I(y_n \leq y_1) \\
&\quad E[G_n(x_n) - G_n(x_0) - G(x_n) + G(x_0) | X_n = x_n, X_1 = x_1] dH(x_1, y_1) \\
&= K^* \left(\frac{G(x_n) - G(x_0)}{a_n} \right) \iint K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 I(y_n \leq y_1) \\
&\quad \frac{1}{n} [I(x_n > x_0) + I(x_1 \leq x_n) - I(x_1 \leq x_0) - 2(G(x_n) - G(x_0))] dH(x_1, y_1) \\
&\quad \text{by (4.4.25),}
\end{aligned}$$

$$\begin{aligned}
& E(t_{n1} | X_n = x_n, Y_n = y_n) \\
&= K_2 \left(\frac{G(x_n) - G(x_0)}{a_n}, \frac{F_{x_0}(y_n) - \lambda}{a_n} \right) y_n \iint K^* \left(\frac{G(x_1) - G(x_0)}{a_n} \right) I(y_1 \leq y_n) \\
&\quad E[G_n(x_1) - G_n(x_0) - G(x_1) + G(x_0) | X_n = x_n, X_1 = x_1] \\
&\quad dH(x_1, y_1) \\
&= K_2 \left(\frac{G(x_n) - G(x_0)}{a_n}, \frac{F_{x_0}(y_n) - \lambda}{a_n} \right) y_n \iint K^* \left(\frac{G(x_1) - G(x_0)}{a_n} \right) I(y_1 \leq y_n) \\
&\quad \frac{1}{n} [I(x_1 > x_0) + I(x_n \leq x_1) - I(x_n \leq x_0) - 2(G(x_1) - G(x_0))] dH(x_1, y_1) \\
&\quad \text{by (4.4.25),}
\end{aligned}$$

$$\begin{aligned}
& E(t_{11} | X_n = x_n, Y_n = y_n) \\
&= \iint K^* \left(\frac{G(x_1) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 \\
&\quad E[G_n(x_1) - G_n(x_0) - G(x_1) + G(x_0) | X_n = x_n, X_1 = x_1] dH(x_1, y_1)
\end{aligned}$$

$$= \iint K^* \left(\frac{G(x_1) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1$$

$$\frac{1}{n} [I(x_1 > x_0) + I(x_n \leq x_1) - I(x_n \leq x_0) - 2(G(x_1) - G(x_0))]$$

$$dH(x_1, y_1)$$

and

$$E(t_{12} | X_n = x_n, Y_n = y_n)$$

$$= \iiint K^* \left(\frac{G(x_2) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 I(y_2 \leq y_1)$$

$$E[G_n(x_2) - G_n(x_0) - G(x_2) + G(x_0) | X_n = x_n, X_1 = x_1, X_2 = x_2]$$

$$dH(x_1, y_1) dH(x_2, y_2)$$

$$= \iiint K^* \left(\frac{G(x_2) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 I(y_2 \leq y_1)$$

$$\frac{1}{n} [I(x_2 > x_0) + I(x_n \leq x_2) + I(x_1 \leq x_2) - I(x_n \leq x_0) - I(x_1 \leq x_0) - 3(G(x_2) - G(x_0))]$$

$$dH(x_1, y_1) dH(x_2, y_2), \text{ by (4.4.26).}$$

Now, as before using the above expressions in (5.1.23), to obtain the expression for

$E(I_{32} | X_n, Y_n)$ and observing that the same expression with (X_n, Y_n) replaced by (X_j, Y_j) gives $E(I_{32} | X_j, Y_j)$ for $j = 1, \dots, n-1$, it follows from (5.1.22) that

$$(5.1.24) \quad \hat{I}_{32} = (n^3 a_n^5)^{-1} \sum K^* \left(\frac{G(X_1) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(X_1) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_1) - \lambda}{a_n} \right) Y_1$$

$$[I(X_1 > x_0) - G(X_1) + G(x_0)]$$

continued

$$\begin{aligned}
& + (n-1)(n^3 a_n^5)^{-1} \int K^* \left(\frac{G(x_1) - G(x_0)}{a_n} \right) \int K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) \\
& \quad y_1 I(y_1 \leq y_1) [I(X_1 > x_0) + I(X_1 \geq x_1) - I(x_1 \leq x_0) \\
& \quad \quad - 2(G(x_1) - G(x_0))] dH(x_1, y_1) \\
& + (n-1)(n^3 a_n^5)^{-1} \int K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 \int K^* \left(\frac{G(x_1) - G(x_0)}{a_n} \right) I(y_1 \geq y_1) \\
& \quad [I(X_1 \leq x_1) - I(X_1 \leq x_0) - (G(x_1) - G(x_0))] dH(x_1, y_1) \\
& + (n-1)(n^3 a_n^5)^{-1} \int \int K^* \left(\frac{G(x_1) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 \\
& \quad [I(X_1 \leq x_1) - I(X_1 \leq x_0) - (G(x_1) - G(x_0))] dH(x_1, y_1) \\
& + (n-1)(n-2)(n^3 a_n^5)^{-1} \int \int \int K^* \left(\frac{G(x_2) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) \\
& \quad y_1 I(y_2 \leq y_1) [I(X_1 \leq x_2) - I(X_1 \leq x_0) - (G(x_2) - G(x_0))] \\
& \quad \quad dH(x_1, y_1) dH(x_2, y_2) \\
& + C_4
\end{aligned}$$

where C_4 is the constant term.

Now (5.1.11), (5.1.17), (5.1.21) and (5.1.24) in (5.1.10) gives (5.1.9).

This completes the proof of the lemma. $\square$

5.2 ASYMPTOTIC EQUIVALENCE OF $S_{n, x_0}(\lambda)$ and $\hat{S}_{n, x_0}(\lambda)$. In this section we

prove that $n^{1/2} a_n S_{n, x_0}(\lambda)$ is asymptotically equivalent in mean square to

$$n^{1/2} a_n \hat{S}_{n, x_0}(\lambda).$$

By lemma 2.1.1 we have

$$(5.2.1) \quad E[n^{1/2}a_n(S_{n,x_0}(\lambda) - \hat{S}_{n,x_0}(\lambda))]^2 = \text{Var}[n^{1/2}a_n S_{n,x_0}(\lambda)] - \text{Var}[n^{1/2}a_n \hat{S}_{n,x_0}(\lambda)]$$

and by (4.4.104) we have

$$\lim_{n \rightarrow \infty} \text{Var}[n^{1/2}a_n S_{n,x_0}(\lambda)] = \sigma_0^2(x_0, \lambda)$$

where $\sigma_0^2(x_0, \lambda)$ is as given by (5.1.3).

Hence to prove the asymptotic equivalence of $n^{1/2}a_n S_{n,x_0}(\lambda)$ and $n^{1/2}a_n \hat{S}_{n,x_0}(\lambda)$

all we need to show is that $\lim_{n \rightarrow \infty} \text{Var}[n^{1/2}a_n \hat{S}_{n,x_0}(\lambda)] = \sigma_0^2(x_0, \lambda)$.

First observe that since $K^*, K^*, K(\dots), K_1'(\dots), K_2'(\dots)$ are all bounded, $E(Y)^2 < \infty$ and $na_n^4 \rightarrow \infty$, from (5.1.15) we see that

$$(5.2.2) \quad \text{Var}(n^{1/2}a_n S_2) \leq \frac{n^2 a_n^2}{n^4 a_n^6} E\left[K_1' \left(\frac{G(X_1) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_1) - \lambda}{a_n} \right) \right. \\ \left. [I(X_1 > x_0) - G(X_1) + G(x_0)] Y_1 \right]^2 \\ = O\left(\frac{1}{n^2 a_n^4}\right), \text{ which goes to zero as } n \rightarrow \infty,$$

$$(5.2.3) \quad \text{Var}(n^{1/2}a_n S_4) \leq \frac{n^2 a_n^2}{n^4 a_n^8} E\left[K^* \left(\frac{G(X_1) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(X_1) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_1) - \lambda}{a_n} \right) Y_1 \right]^2 \\ = O\left(\frac{1}{n^2 a_n^6}\right), \text{ which goes to zero as } n \rightarrow \infty,$$

$$(5.2.4) \quad \text{Var}(n^{1/2}a_n S_7) \leq \frac{n^2 a_n^2}{n^6 a_n^{10}} E\left[K^* \left(\frac{G(X_1) - G(x_0)}{a_n} \right) K_2' \left(\frac{G(X_1) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_1) - \lambda}{a_n} \right) Y_1 \right. \\ \left. [I(X_1 > x_0) - G(X_1) + G(x_0)] \right]^2 \\ = O\left(\frac{1}{n^4 a_n^8}\right), \text{ which goes to zero as } n \rightarrow \infty,$$

$$\begin{aligned}
 (5.2.5) \quad \text{Var}(n^{1/2} a_n S_8) &\leq \frac{n^2 a_n^2 (n-1)^2}{n^6 a_n^{10}} E \left[K^* \left(\frac{G(X_1) - G(x_0)}{a_n} \right) \int K_2 \left(\frac{G(x) - G(x_0)}{a_n}, \frac{F_{x_0}(y) - \lambda}{a_n} \right) \right. \\
 &\quad \left. y [I(X_1 > x) - I(X_1 \leq x_0) + I(x > x_0) - 2(G(X_1) - G(x_0))] \right. \\
 &\quad \left. I(Y_1 \leq y) dH(x, y) \right]^2 \\
 &= O\left(\frac{1}{n^2 a_n^8}\right), \text{ which goes to zero as } n \rightarrow \infty,
 \end{aligned}$$

$$\begin{aligned}
 (5.2.6) \quad \text{Var}(n^{1/2} a_n S_9) &\leq \frac{n^2 a_n^2 (n-1)^2}{n^6 a_n^{10}} E \left[K_2 \left(\frac{G(X_1) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_1) - \lambda}{a_n} \right) Y_1 \right. \\
 &\quad \left. \int K^* \left(\frac{G(x) - G(x_0)}{a_n} \right) I(Y_1 > y) [I(X_1 \leq x) - I(X_1 \leq x_0) \right. \right. \\
 &\quad \left. \left. + I(X > x_0) - 2(G(x) - G(x_0))] dH(x, y) \right]^2 \\
 &= O\left(\frac{1}{n^2 a_n^8}\right), \text{ which goes to zero as } n \rightarrow \infty,
 \end{aligned}$$

and

$$\begin{aligned}
 (5.2.7) \quad \text{Var}(n^{1/2} a_n S_{10}) &= \frac{n^2 a_n^2 (n-1)^2}{n^6 a_n^{10}} E \left[\int K^* \left(\frac{G(x) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(x) - G(x_0)}{a_n}, \frac{F_{x_0}(y) - \lambda}{a_n} \right) \right. \\
 &\quad \left. [I(X_1 \leq x) - I(X_1 \leq x_0) - (G(x) - G(x_0))] dH(x, y) \right]^2 \\
 &= O\left(\frac{1}{n^2 a_n^8}\right), \text{ which goes to zero as } n \rightarrow \infty.
 \end{aligned}$$

LEMMA 5.2.1. Let S_3 and S_{11} be as given in (5.1.9). Suppose assumptions (4.1.11) to (4.1.13) on the joint distribution H are satisfied and assumptions (4.1.3) to (4.1.6) on K and K^* are satisfied with $v = 1$, $v^* = 1$. Then for any sequence $\{a_n\}$ of positive numbers with $a_n \rightarrow 0$ and $na_n^4 \rightarrow \infty$, we have

$$(a) \lim_{n \rightarrow \infty} \text{Var}(n^{1/2} a_n S_3) = 0.$$

(5.2.8)

$$(b) \lim_{n \rightarrow \infty} \text{Var}(n^{1/2} a_n S_{11}) = 0.$$

PROOF.

$$(a) \text{ Since } S_3 = \frac{n-1}{n^2 a_n^3} \left[\iint K_1' \left(\frac{G(x)-G(x_0)}{a_n}, \frac{F_{x_0}(y_1)-\lambda}{a_n} \right) [I(X_1 \leq x) - I(X_1 \leq x_0) - (G(x)-G(x_0))] y dH(x, y), \right.$$

$$\text{we have } E(S_3) = 0$$

and

$$\begin{aligned} \text{Var}(n^{1/2} a_n S_3) &= \frac{n^2 a_n^2 (n-1)^2}{n^4 a_n^6} E \left\{ \iint K_1' \left(\frac{G(x)-G(x_0)}{a_n}, \frac{F_{x_0}(y_1)-\lambda}{a_n} \right) [I(X_1 \leq x) - I(X_1 \leq x_0) - (G(x)-G(x_0))] y dH(x, y) \right\}^2 \\ &= \frac{(n-1)^2}{n^2 a_n^4} \iiint K_1' \left(\frac{G(x_1)-G(x_0)}{a_n}, \frac{F_{x_0}(y_1)-\lambda}{a_n} \right) K_1' \left(\frac{G(x_2)-G(x_0)}{a_n}, \frac{F_{x_0}(y_2)-\lambda}{a_n} \right) y_1 y_2 \\ &\quad E \{ [I(X_1 \leq x_1) - I(X_1 \leq x_0) - (G(x_1)-G(x_0))] \\ &\quad [I(X_1 \leq x_2) - I(X_1 \leq x_0) - (G(x_2)-G(x_0))] \} dH(x_1, y_1) dH(x_2, y_2) \\ &= \frac{(n-1)^2}{n^2 a_n^4} \iiint K_1' \left(\frac{G(x_1)-G(x_0)}{a_n}, \frac{F_{x_0}(y_1)-\lambda}{a_n} \right) K_1' \left(\frac{G(x_2)-G(x_0)}{a_n}, \frac{F_{x_0}(y_2)-\lambda}{a_n} \right) y_1 y_2 \\ &\quad [G(x_1 \wedge x_2) - G(x_1 \wedge x_0) - G(x_2 \wedge x_0) + G(x_0) - (G(x_1) - G(x_0))(G(x_2) - G(x_0))] \\ &\quad dH(x_1, y_1) dH(x_2, y_2). \end{aligned}$$

Observe that the integral in the last expression is the same as that in (4.4.17) from which it follows that it is of order $O(a_n^5)$. Hence

$$\text{Var}(n^{1/2} a_n S_3) = \frac{(n-1)^2}{n^2 a_n^4} O(a_n^5) = O(a_n),$$

and since $a_n \rightarrow 0$ as $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} \text{Var}(n^{1/2} a_n S_3) = 0.$$

(b) From (5.1.9) we have

$$S_{11} = \frac{(n-1)(n-2)}{n^3 a_n^5} \sum \gamma_1, \text{ where}$$

$$\gamma_1 = \iiint K^* \left(\frac{G(x_2) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) \\ [I(X_1 \leq x_2) - I(X_1 \leq x_0) - (G(x_2) - G(x_0))] \\ y_1 I(y_2 \leq y_1) dH(x_1, y_1) dH(x_1, y_1) dH(x_2, y_2).$$

Therefore

$$\text{Var}(n^{1/2} a_n S_{11}) = \frac{(n-1)^2 (n-2)^2}{n^4 a_n^8} \text{Var}(\gamma_1)$$

and

$$\text{Var}(\gamma_1) = \iiint K^* \left(\frac{G(x_2) - G(x_0)}{a_n} \right) K^* \left(\frac{G(x_4) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) \\ K_2 \left(\frac{G(x_3) - G(x_0)}{a_n}, \frac{F_{x_0}(y_3) - \lambda}{a_n} \right) y_1 y_3 I(y_2 \leq y_1) I(y_4 \leq y_3) \\ [G(x_2 \wedge x_4) - G(x_2 \wedge x_0) - G(x_4 \wedge x_0) + G(x_0) - (G(x_2) - G(x_0))(G(x_4) - G(x_0))] \\ dH(x_1, y_1) dH(x_2, y_2) dH(x_3, y_3) dH(x_4, y_4).$$

But the last integral is similar to that in (4.4.42) and it follows from the computations therein, that

$$\text{Var}(\gamma_1) = O(a_n^9)$$

and consequently we have

$$\text{Var}(n^{1/2} a_n S_{11}) = O(a_n)$$

which goes to zero as $n \rightarrow \infty$. $\square$

LEMMA 5.2.2 Suppose assumptions (4.1.11) to (4.1.13) on the joint distribution H are satisfied and S_5 is as in (5.1.5). Then

$$\text{Var}(n^{1/2} a_n S_5) \rightarrow \sigma_1^2(x_0, \lambda), \text{ as } n \rightarrow \infty$$

where $\sigma_1^2(x_0, \lambda) = (F_{x_0}^{-1}(\lambda))^2 \iint K^*(u) K_2^2(v) du dv$, as given by (4.3.45).

PROOF. From (5.1.9) we have

$$S_5 = \frac{n-1}{n^2 a_n^4} \sum \alpha_1, \text{ where}$$

$$(5.2.9) \quad \alpha_1 = K^* \left(\frac{G(X_1) - G(x_0)}{a_n} \right) \iint K_2^* \left(\frac{G(x) - G(x_0)}{a_n} \right), \frac{F_{x_0}(y) - \lambda}{a_n} y I(Y_1 \leq y) dH(x, y).$$

Therefore,

$$(5.2.10) \quad \text{Var}(n^{1/2} a_n S_5) = \frac{(n-1)^2}{n^2 a_n^6} \text{Var}(\alpha_1)$$

and

$$(5.2.11) \quad E(\alpha_1^2) = \iiint K^{*2} \left(\frac{G(x_3) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) \\ K_2 \left(\frac{G(x_2) - G(x_0)}{a_n}, \frac{F_{x_0}(y_2) - \lambda}{a_n} \right) y_1 y_2 I(y_3 \leq y_1 \wedge y_2) dH(x_1, y_1) \\ dH(x_2, y_2) dH(x_3, y_3),$$

$$(5.2.12) \quad E^2(\alpha_1) = \iiint K^{*2} \left(\frac{G(x_2) - G(x_0)}{a_n} \right) K^{*2} \left(\frac{G(x_4) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) \\ K_2 \left(\frac{G(x_3) - G(x_0)}{a_n}, \frac{F_{x_0}(y_3) - \lambda}{a_n} \right) y_1 y_3 I(y_2 \leq y_1) I(y_4 \leq y_3) \\ dH(x_1, y_1) dH(x_2, y_2) dH(x_3, y_3) dH(x_4, y_4).$$

Observe that the integrals in (5.2.11) and (5.2.12) are exactly those in

(4.4.59) corresponding to $E(l_{13}l_{23})$ and (4.4.54) corresponding to

$E(l_{13})E(l_{23})$ respectively. Hence, it follows from (4.4.64) that

$$\text{Var}(\alpha_1) = \text{Cov}(l_{13}, l_{23}) \\ = 2(F_{x_0}^{-1}(\lambda))^2 a_n^6 \iiint v_1 I(v_1 \leq v_2) K_2'(u_1, v_1) K_2'(u_2, v_2) (K^{*2}(u_3) du_3) \\ du_1 dv_1 du_2 dv_2 \\ + O(a_n^7).$$

$$= (F_{x_0}^{-1}(\lambda))^2 a_n^6 \iint K^{*2}(u) K_2^2(v) du dv + O(a_n^7),$$

by (4.4.84), and consequently we have from (5.2.10)

$$\text{Var}(n^{1/2} a_n S_5) = \frac{(n-1)^2}{n^2} (F_{x_0}^{-1}(\lambda))^2 \iint K^{*2}(u) K_2^2(v) du dv \\ + O(a_n).$$

Therefore

$$(5.2.13) \quad \lim_{n \rightarrow \infty} \text{Var}(n^{1/2} a_n S_5) = (F_{x_0}^{-1}(\lambda))^2 \iint K^{*2}(u) K_2^2(v) du dv. \quad \square$$

LEMMA 5.2.3. Under the hypothesis of lemma 5.2.1, we have

$$(5.2.14) \quad \lim_{n \rightarrow \infty} \text{Var}[n^{1/2} a_n (S_6 - S_{12})] = 0.$$

PROOF. From (5.1.9) we have $S_6 = \frac{n-1}{n^2 a_n} \sum \beta_1$, where

$$(5.2.15) \quad \beta_1 = K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 \iint K^* \left(\frac{G(x) - G(x_0)}{a_n} \right) I(y_1 \geq y) dH(x, y),$$

so that

$$(5.2.16) \quad \text{Var}(n^{1/2} a_n S_6) = \frac{(n-1)^2}{n^2 a_n} \text{Var}(\beta_1)$$

and

$$(5.2.17) \quad E(\beta_1^2) = \iiint \iint K_2^2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) K^* \left(\frac{G(x_2) - G(x_0)}{a_n} \right) \\ K^* \left(\frac{G(x_3) - G(x_0)}{a_n} \right) y_1^2 I(y_2 \leq y_1) I(y_3 \leq y_1) dH(x_1, y_1) dH(x_2, y_2) \\ dH(x_3, y_3),$$

$$\begin{aligned}
 (5.2.18) \quad E^2(\beta_1) = & \iiint K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) K^* \left(\frac{G(x_3) - G(x_0)}{a_n} \right. \\
 & \left. K_2 \left(\frac{G(x_2) - G(x_0)}{a_n}, \frac{F_{x_0}(y_2) - \lambda}{a_n} \right) K^* \left(\frac{G(x_4) - G(x_0)}{a_n} \right) \right. \\
 & \left. y_1 y_2 I(y_3 \leq y_1) I(y_4 \leq y_2) dH(x_1, y_1) dH(x_2, y_2) \right. \\
 & \left. dH(x_3, y_3) dH(x_4, y_4) \right).
 \end{aligned}$$

Observe that the integral in (5.2.18) is same as that in (4.4.54) from where it follows that

$$(5.2.19) \quad E^2(\beta_1) = O(a_n^7).$$

We also observe that the integral in (5.2.17) is the same as that in (4.4.56) and hence it follows from (4.4.56), (5.2.19) and (5.2.16) that

$$(5.2.20) \quad \text{Var}(n^{1/2} a_n S_6)$$

$$\begin{aligned}
 &= \frac{(n-1)^2}{n^2 a_n^2} \iint K_2'^2(u_1, v_1) \lambda^2 (F_{x_0}^{-1}(\lambda))^2 du_1 dv_1 \\
 &+ \frac{(n-1)^2}{n^2 a_n^2} \iint K_2'^2(u_1, v_1) [2\lambda^2 F_{x_0}^{-1}(\lambda) v_1 D_1 + 2\lambda (F_{x_0}^{-1}(\lambda))^2 v_1 + \lambda^2 (F_{x_0}^{-1}(\lambda))^2 \\
 &\quad u_1 D_6] du_1 dv_1 \\
 &+ \frac{(n-1)^2}{n^2} \iint K_2'^2(u_1, v_1) [2\lambda^2 F_{x_0}^{-1}(\lambda) v_1^2 D_2 + \lambda^2 v_1^2 D_1^2 + \lambda^2 (F_{x_0}^{-1}(\lambda))^2 \\
 &\quad (u_1^2 D_7 + u_1 v_1 D_8) + 4\lambda F_{x_0}^{-1}(\lambda) v_1^2 D_1 + 2\lambda^2 F_{x_0}^{-1}(\lambda) v_1 u_1 D_6 \\
 &\quad + (F_{x_0}^{-1}(\lambda))^2 v_1^2 + 2\lambda (F_{x_0}^{-1}(\lambda))^2 v_1 u_1 D_6] \\
 &+ \frac{(n-1)^2}{n^2} \iint K_2'^2(u_1, v_1) 2\lambda (F_{x_0}^{-1}(\lambda))^2 D_4 du_1 dv_1 (f u_2^2 K^*(u_2) du_2) \\
 &+ O(a_n).
 \end{aligned}$$

Since $S_{12} \equiv I_{30}$, we have $\text{Var}(n^{1/2} a_n S_{12})$ as given by (4.4.75).

We observe that

$$\begin{aligned}
 (5.2.21) \quad \text{Cov}(n^{1/2} a_n S_6, n^{1/2} a_n S_{12}) \\
 &= n a_n^2 \text{Cov}\left(\frac{n-1}{n^2 a_n^4} \sum \beta_1, \frac{1}{n a_n^3} \sum p_j\right) \\
 &= \frac{(n-1)}{n a_n^5} \text{Cov}(\beta_1, p_1),
 \end{aligned}$$

where β_1 's are defined by (5.2.15) and p_j 's by (4.4.72).

Now,

$$\begin{aligned}
 (5.2.22) \quad E(\beta_1 p_1) &= E\left[K_2' \left(\frac{G(X_1) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_1) - \lambda}{a_n} \right) Y_1 \int K^* \left(\frac{G(x) - G(x_0)}{a_n} \right) I(Y_1 \geq y) dH(x, y) \right. \\
 &\quad \left. \times K_2' \left(\frac{G(X_1) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_1) - \lambda}{a_n} \right) Y_1 F_{x_0}(Y_1) \right] \\
 &= \iiint K_2'^2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) K^* \left(\frac{G(x_2) - G(x_0)}{a_n} \right) y_1^2 I(y_2 \leq y_1) \\
 &\quad F_{x_0}(y_1) dH(x_1, y_1) dH(x_2, y_2)
 \end{aligned}$$

and

$$\begin{aligned}
 (5.2.23) \quad E(\beta_1) E(p_1) &= \iiint K^* \left(\frac{G(x_2) - G(x_0)}{a_n} \right) K_2' \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) \\
 &\quad K_2' \left(\frac{G(x_3) - G(x_0)}{a_n}, \frac{F_{x_0}(y_3) - \lambda}{a_n} \right) y_1 y_3 I(y_2 \leq y_1) F_{x_0}(y_3) \\
 &\quad dH(x_1, y_1) dH(x_2, y_2) dH(x_3, y_3).
 \end{aligned}$$

Since the integrals in (5.2.22) and (5.2.23) are the same as those in (4.4.77) and (4.4.80) respectively, it follows from (4.4.78), (4.4.81) and (5.2.21) that

$$\begin{aligned}
 (5.2.24) \quad & \text{Cov}(n^{1/2}a_n S_6, n^{1/2}a_n S_{12}) \\
 &= \frac{n-1}{na_n^2} \iint K_2'(u_1, v_1) \lambda^2 (F_{x_0}^{-1}(\lambda))^2 du_1 dv_1 \\
 &+ \frac{n-1}{na_n} \iint K_2'(u_1, v_1) [2\lambda^2 F_{x_0}^{-1}(\lambda) v_1 D_1 + 2\lambda (F_{x_0}^{-1}(\lambda))^2 v_1 + \lambda^2 (F_{x_0}^{-1}(\lambda))^2 u_1 D_6] \\
 &\quad du_1 dv_1 \\
 &+ \frac{n-1}{n} \iint K_2'(u_1, v_1) [2\lambda^2 F_{x_0}^{-1}(\lambda) v_1^2 D_2 + \lambda^2 v_1^2 D_1^2 + \lambda^2 (F_{x_0}^{-1}(\lambda))^2 \\
 &\quad (u_1^2 D_7 + u_1 v_1 D_8) + 4\lambda F_{x_0}^{-1}(\lambda) v_1^2 D_1 + 2\lambda^2 F_{x_0}^{-1}(\lambda) v_1 u_1 D_1 D_6 \\
 &\quad + (F_{x_0}^{-1}(\lambda))^2 v_1^2 + 2\lambda (F_{x_0}^{-1}(\lambda))^2 v_1 u_1 D_6] du_1 dv_1 \\
 &+ \frac{n-1}{n} \iint K_2'(u_1, v_1) \lambda (F_{x_0}^{-1}(\lambda))^2 D_4 du_1 dv_1 (\int u_2^2 K^*(u_2) du_2) \\
 &+ O(a_n).
 \end{aligned}$$

$$\begin{aligned}
 \text{Since } \text{Var}[n^{1/2}a_n(S_6 - S_{12})] &= \text{Var}(n^{1/2}a_n S_6) + \text{Var}(n^{1/2}a_n S_{12}) \\
 &\quad - 2\text{Cov}(n^{1/2}a_n S_6, n^{1/2}a_n S_{12}),
 \end{aligned}$$

It follows from (5.2.20), (4.4.75) and (5.2.24) that

$$\text{Var}[n^{1/2}a_n(S_6 - S_{12})] = O\left(\frac{1}{n^2 a_n^2}\right) + O\left(\frac{1}{n^2 a_n}\right) + O\left(\frac{1}{n^2}\right) + O(a_n).$$

The lemma now follows immediately on letting $n \rightarrow \infty$ in the above expression, since $na_n^4 \rightarrow \infty$ and $a_n \rightarrow 0$ as $n \rightarrow \infty$. $\square$

LEMMA 5.2.4 Under the hypothesis of lemma 5.2.1, we have

$$(5.2.25) \quad \lim_{n \rightarrow \infty} \text{Var}(n^{1/2} a_n \hat{S}_{n, x_0}(\lambda)) = \sigma_0^2(x_0, \lambda)$$

where $\sigma_0^2(x_0, \lambda) = (F_{x_0}^{-1}(\lambda))^2 \iint [K(u, v) - K^*(u)K_2(v)]^2 du dv$, as given by (5.1.3).

PROOF. From (5.2.1) to (5.2.7), lemmas 5.2.1 to 5.2.3, clearly it suffices to confine attention to the terms $n^{1/2} a_n S_1$ and $n^{1/2} a_n S_5$ in order to find the limiting expression for $\text{Var}(n^{1/2} a_n \hat{S}_{n, x_0}(\lambda))$.

Since $S_1 \equiv I_1$, from lemma 4.4.3, we have

$$(5.2.26) \quad \lim_{n \rightarrow \infty} \text{Var}(n^{1/2} a_n S_1) = (F_{x_0}^{-1}(\lambda))^2 \iint K^2(u, v) du dv.$$

We also have from lemma 5.2.2

$$(5.2.27) \quad \lim_{n \rightarrow \infty} \text{Var}(n^{1/2} a_n S_5) = (F_{x_0}^{-1}(\lambda))^2 \iint K^*(u) K_2^2(v) du dv.$$

Now, observe that

$$\begin{aligned} (5.2.28) \quad & \text{Cov}(n^{1/2} a_n S_1, n^{1/2} a_n S_5) \\ &= n a_n^2 \text{Cov}\left(\frac{1}{n a_n^2} \sum q_i, \frac{n-1}{n^2 a_n^4} \sum \alpha_j\right) \\ &= \frac{n-1}{n a_n^4} \text{Cov}(q_1, \alpha_1) \end{aligned}$$

$$\text{where } \alpha_1 = K \left(\frac{G(X_1) - G(x_0)}{a_n} \right) \iint K_2 \left(\frac{G(x) - G(x_0)}{a_n} \right) \frac{F_{x_0}^{-1}(y) - \lambda}{a_n} y I(Y_1 \leq y) dH(x, y),$$

as defined by (5.2.9) and

$$q_1 = K\left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n}\right) y_1, \text{ as defined by (4.4.86).}$$

Therefore

$$(5.2.29) \quad E(q_1, \alpha_1) = \iiint K^*\left(\frac{G(x_2) - G(x_0)}{a_n}, \frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n}\right) \\ K\left(\frac{G(x_2) - G(x_0)}{a_n}, \frac{F_{x_0}(y_2) - \lambda}{a_n}\right) y_1 y_2 I(y_2 \leq y_1) dH(x_1, y_1) \\ dH(x_2, y_2)$$

and

$$(5.2.30) \quad E(q_1)E(\alpha_1) = \iiint K^*\left(\frac{G(x_2) - G(x_0)}{a_n}, \frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n}\right) \\ K\left(\frac{G(x_3) - G(x_0)}{a_n}, \frac{F_{x_0}(y_3) - \lambda}{a_n}\right) y_1 y_3 I(y_2 \leq y_1) \\ dH(x_1, y_1) dH(x_2, y_2) dH(x_3, y_3).$$

As before, once again observe that the integrals in (5.2.29) and (5.2.30) are exactly the same as those in (4.4.93) and (4.4.95) respectively.

Consequently, it follows from (4.4.94) and (4.4.96) that

$$(5.2.31) \quad \text{Cov}(q_1, \alpha_1) = a_n^4 \iiint K^*(u_1) K_2'(u_2, v_2) K(u_1, v_1) I(v_1 \leq v_2) (F_{x_0}^{-1}(\lambda))^2 du_1 dv_1 du_2 dv_2 \\ + O(a_n^5) \\ = -a_n^4 (F_{x_0}^{-1}(\lambda))^2 \iint K^*(u) K_2(v) K(u, v) du dv + O(a_n^5), \text{ by (4.4.103)}$$

Hence from (5.2.28) and (5.2.31) we have

$$\text{Cov}(n^{1/2}a_n S_1, n^{1/2}a_n S_5) = -\frac{(n-1)}{n} (F_{x_0}^{-1}(\lambda))^2 \iint K^*(u) K_2(v) K(u, v) du dv + O(a_n)$$

and therefore

$$(5.2.32) \quad \lim_{n \rightarrow \infty} \text{Cov}(n^{1/2}a_n S_1, n^{1/2}a_n S_5) = -(F_{x_0}^{-1}(\lambda))^2 \iint K^*(u) K_2(v) K(u, v) du dv.$$

Now from equations (5.2.1) to (5.2.6) and lemmas 5.2.1 to 5.2.3 we have

$$(5.2.33) \quad \lim_{n \rightarrow \infty} \text{Var}(n^{1/2}a_n \hat{S}_{n, x_0}(\lambda)) = \lim_{n \rightarrow \infty} \text{Var}(n^{1/2}a_n S_1 + n^{1/2}a_n S_5),$$

and the lemma now follows from (5.2.26), (5.2.27) and (5.2.32). $\square$

Because of (4.4.104) and (5.2.25), by taking the limit as $n \rightarrow \infty$ in (5.2.1) we have

$$\lim_{n \rightarrow \infty} E[n^{1/2}a_n (S_{n, x_0}(\lambda) - \hat{S}_{n, x_0}(\lambda))]^2 = 0.$$

i.e. $n^{1/2}a_n S_{n, x_0}(\lambda)$ and $n^{1/2}a_n \hat{S}_{n, x_0}(\lambda)$ are asymptotically equivalent in mean square.

5.3. ASYMPTOTIC NORMALITY OF $Q_{n, x_0}(\lambda)$. In this section we give proofs of Theorem 5.1.1 and Corollary 5.1.1.

PROOF OF THEOREM 5.1.1. First observe that lemma 5.1.1 implies that

$n^{1/2}a_n (Q_{n, x_0}(\lambda) - \bar{Q}_{n, x_0}(\lambda))$ has the same asymptotic distribution as

$n^{1/2}a_n (S_{n, x_0}(\lambda) - \bar{Q}_{n, x_0}(\lambda))$. But $n^{1/2}a_n (\hat{S}_{n, x_0}(\lambda) - \bar{Q}_{n, x_0}(\lambda))$ and

$n^{1/2}a_n (S_{n, x_0}(\lambda) - \bar{Q}_{n, x_0}(\lambda))$ are asymptotically equivalent in mean square, which

implies that $n^{1/2}a_n(Q_{n,x_0}(\lambda) - \bar{Q}_{n,x_0}(\lambda))$ has the same asymptotic distribution as $n^{1/2}a_n(\hat{S}_{n,x_0}(\lambda) - \bar{Q}_{n,x_0}(\lambda))$.

Now, we further show that $n^{1/2}a_n(\hat{S}_{n,x_0}(\lambda) - \bar{Q}_{n,x_0}(\lambda))$ has indeed the same asymptotic distribution as $n^{1/2}a_n(S_1 + S_5 - \bar{Q}_{n,x_0}(\lambda))$ where S_1 and S_5 are as defined in (5.1.9), by showing that $[n^{1/2}a_n\hat{S}_{n,x_0}(\lambda) - n^{1/2}a_n(S_1 + S_5)]$ converges to zero in probability as $n \rightarrow \infty$. From (5.1.9) we can write

$$\begin{aligned} n^{1/2}a_n\hat{S}_{n,x_0}(\lambda) - n^{1/2}a_n(S_1 + S_5) \\ = n^{1/2}a_n[S_2 + S_3 + S_4 + (S_6 - S_{12}) + S_7 + S_8 + S_9 + S_{10} + S_{11}]. \end{aligned}$$

From equations (5.2.1) to (5.2.7), lemma 5.2.1 and lemma 5.2.3, we already have $\text{Var}(n^{1/2}a_n S_2)$, $\text{Var}(n^{1/2}a_n S_3)$, $\text{Var}(n^{1/2}a_n S_4)$, $\text{Var}(n^{1/2}a_n (S_6 - S_{12}))$, $\text{Var}(n^{1/2}a_n S_7)$, ..., $\text{Var}(n^{1/2}a_n S_{11})$ all going to zero as $n \rightarrow \infty$. Hence it suffices to show that the corresponding expectations also go to zero as $n \rightarrow \infty$. Using the expressions for S_i 's from (5.1.9), first note that

$$E(S_3) = 0 = E(S_{10}) = E(S_{11}).$$

Now

$$E(n^{1/2}a_n S_2) = (n^{1/2}a_n^2)^{-1} \iint K'_1 \left(\frac{G(x) - G(x_0)}{a_n}, \frac{F_{x_0}(y) - \lambda}{a_n} \right) [I(x > x_0) - (G(x) - G(x_0))] y dH(x, y)$$

which goes to zero as $n \rightarrow \infty$, since K'_1 is bounded $E|Y| < \infty$ and $na_n^4 \rightarrow \infty$, and similarly

$$E(n^{1/2}a_n S_7) = O\left(\frac{1}{n^{3/2}a_n^4}\right), \text{ goes to zero as } n \rightarrow \infty.$$

$$E(n^{1/2} a_n S_4) = (n^{1/2} a_n^3)^{-1} \iint K^* \left(\frac{G(x) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(x) - G(x_0)}{a_n}, \frac{F_{x_0}(y) - \lambda}{a_n} \right) y dH(x, y).$$

By transformation of variables (4.4.13) and application of lemma 4.1.1,

for $n > n_0$ the above integral may be written as

$$\begin{aligned} &= a_n^2 \int_{-1}^1 \int_{-1}^1 K^*(u) K_2'(u, v) [F_{x_0}^{-1}(\lambda) + a_n v D_1 + a_n^2 v^2 D_2 + O(a_n^3)] \\ &\quad [1 + a_n u D_6 + a_n^2 u^2 D_7 + a_n^2 u v D_8 + O(a_n^3)] du dv \\ &= O(a_n^2). \end{aligned}$$

Hence we have $E(n^{1/2} a_n S_4) = O(n^{-1/2} a_n^{-1})$ which goes to zero as $n \rightarrow \infty$.

Proceeding exactly on the same lines as above, we easily see that

$$E(n^{1/2} a_n S_8) = \frac{n-1}{n^{3/2} a_n^4} O(a_n^3) = O\left(\frac{1}{n^{1/2} a_n}\right)$$

and

$$E(n^{1/2} a_n S_9) = \frac{n-1}{n^{3/2} a_n^4} O(a_n^3) = O\left(\frac{1}{n^{1/2} a_n}\right),$$

which go to zero as $n \rightarrow \infty$.

We now consider $E(n^{1/2} a_n (S_6 - S_{12}))$. First we obtain

$$\begin{aligned} E(S_6) &= \frac{n-1}{n a_n^4} \iiint K^* \left(\frac{G(x_2) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 I(y_2 \leq y_1) \\ &\quad dH(x_1, y_1) dH(x_2, y_2) \\ &= \frac{n-1}{n a_n^4} \iiint K^* \left(\frac{G(x_2) - G(x_0)}{a_n} \right) K_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 F_{x_2}(y_1) \\ &\quad dH(x_1, y_1) dG(x_2). \end{aligned}$$

After the transformation of variables (4.4.18) followed by the use of Taylor expansions from lemma 4.1.1, we see that for $n > n_0$

$$\begin{aligned}
 (5.3.1) \quad E(S_6) &= \frac{n-1}{na_n} \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 K^*(u_2) K'_2(u_1, v_1) [F_{x_0}^{-1}(\lambda) + a_n v_1 D_1 + a_n^2 v_1^2 D_2 + O(a_n^3)] \\
 &\quad [\lambda + a_n u_2 D_3 + a_n v_1 + a_n^2 u_2^2 D_4 + a_n u_2 v_1 D_5 + O(a_n^3)] \\
 &\quad [1 + a_n u_1 D_6 + a_n^2 u_1^2 D_7 + a_n^2 u_1 v_1 D_8 + O(a_n^3)] du_1 dv_1 du_2 dv_2 \\
 &= \frac{n-1}{n} \iint K'_2(u_1, v_1) [\lambda v_1 D_1 + F_{x_0}^{-1}(\lambda) v_1 + \lambda F_{x_0}^{-1}(\lambda) u_1 D_6] du_1 dv_1 \\
 &\quad + \frac{n-1}{n} a_n \iint K'_2(u_1, v_1) [\lambda v_1^2 D_2 + \lambda F_{x_0}^{-1}(\lambda) (u_1^2 D_7 + u_1 v_1 D_8) + v_1^2 D_1 + \lambda v_1 u_1 D_6 D_1 \\
 &\quad + F_{x_0}^{-1}(\lambda) v_1 u_1 D_6] du_1 dv_1 \\
 &\quad + O(a_n^2),
 \end{aligned}$$

where the last equality in (5.3.1) follows by expanding in powers of a_n and

utilising the hypothesis $\int_{-1}^1 K^*(u) du = 1$, $\int_{-1}^1 u K^*(u) du = 0$, $\int_{-1}^1 K'_2(u, v) dv = 0$.

Similarly, we find

$$\begin{aligned}
 E(S_{12}) &= \frac{1}{a_n^3} \iint K'_2 \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 F_{x_0}(y_1) dH(x_1, y_1) \\
 &= \frac{1}{a_n} \int_{-1}^1 \int_{-1}^1 K'_2(u_1, v_1) [F_{x_0}^{-1}(\lambda) + a_n v_1 D_1 + a_n^2 v_1^2 D_2 + O(a_n^3)] [\lambda + a_n v_1] \\
 &\quad [1 + a_n u_1 D_6 + a_n^2 u_1^2 D_7 + a_n^2 u_1 v_1 D_8 + O(a_n^3)] du_1 dv_1
 \end{aligned}$$

for $n > n_0$, by using transformation of variables (4.4.13) and applying lemma

4.1.1.

Therefore, we have

$$\begin{aligned}
 (5.3.2) \quad E(S_{12}) = & \iint K'_2(u_1, v_1) [\lambda v_1 D_1 + F_{x_0}^{-1}(\lambda) v_1 + \lambda F_{x_0}^{-1}(\lambda) u_1 D_6] \\
 & + a_n \iint K'_2(u_1, v_1) [\lambda v_1^2 D_2 + \lambda F_{x_0}^{-1}(\lambda) (u_1^2 D_7 + u_1 v_1 D_8) + v_1^2 D_1 + \lambda v_1 u_1 D_1 D_6 \\
 & + F_{x_0}^{-1}(\lambda) v_1 u_1 D_6] du_1 dv_1 \\
 & + O(a_n^2).
 \end{aligned}$$

From (5.3.1) and (5.3.2), we have

$$E(S_6 - S_{12}) = O\left(\frac{1}{n}\right) + O\left(\frac{a_n}{n}\right) + O(a_n^2)$$

and hence

$$E(n^{1/2} a_n (S_6 - S_{12})) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus we have established that $n^{1/2} a_n (\hat{S}_{n, x_0}(\lambda) - \bar{Q}_{n, x_0}(\lambda))$ has the same

asymptotic distribution as that of $n^{1/2} a_n (S_1 + S_5 - \bar{Q}_{n, x_0}(\lambda))$. Now by the

series of implications mentioned in the beginning of the proof it is clear that

$n^{1/2} a_n (Q_{n, x_0}(\lambda) - \bar{Q}_{n, x_0}(\lambda))$ has the same asymptotic distribution as that of

$n^{1/2} a_n (S_1 + S_5 - \bar{Q}_{n, x_0}(\lambda))$ and hence it suffices to prove the asymptotic

normality of the latter. We have

$$\begin{aligned}
& n^{1/2} a_n (S_1 + S_5 - \bar{Q}_{n, x_0}(\lambda)) \\
&= n^{1/2} a_n \left[\frac{1}{n a_n^2} \sum K\left(\frac{G(X_1) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_1) - \lambda}{a_n}\right) Y_1 \right. \\
&\quad + \frac{n-1}{n^2 a_n^4} \sum K^*\left(\frac{G(X_1) - G(x_0)}{a_n}\right) \iint K_2\left(\frac{G(x) - G(x_0)}{a_n}, \frac{F_{x_0}(y) - \lambda}{a_n}\right) y I(Y_1 \leq y) dH(x, y) \\
&\quad \left. - \frac{1}{a_n^2} \iint K\left(\frac{G(x) - G(x_0)}{a_n}, \frac{F_{x_0}(y) - \lambda}{a_n}\right) y dH(x, y) \right]
\end{aligned}$$

which may be rewritten as

$$\begin{aligned}
(5.3.3) \quad T_n &= n^{1/2} a_n (S_1 + S_5 - \bar{Q}_{n, x_0}(\lambda)) \\
&= \sum \left[\frac{1}{n^{1/2} a_n} K\left(\frac{G(X_1) - G(x_0)}{a_n}, \frac{F_{x_0}(Y_1) - \lambda}{a_n}\right) Y_1 \right. \\
&\quad + \frac{n-1}{n^{3/2} a_n^3} K^*\left(\frac{G(X_1) - G(x_0)}{a_n}\right) \iint K_2\left(\frac{G(x) - G(x_0)}{a_n}, \frac{F_{x_0}(y) - \lambda}{a_n}\right) y I(Y_1 \leq y) dH(x, y) \\
&\quad \left. - \frac{1}{n^{1/2} a_n} \iint K\left(\frac{G(x) - G(x_0)}{a_n}, \frac{F_{x_0}(y) - \lambda}{a_n}\right) y dH(x, y) \right] \\
&= \sum [p_n(X_1, Y_1) + q_n(X_1, Y_1) - C_n], \text{ say,}
\end{aligned}$$

where C_n is the non-random term.

Observe that for each n , T_n is a sum of iid rv's with

$$\sigma_n^2 = \text{Var}[n^{1/2} a_n(S_1 + S_5 - \bar{q}_{n,x_0}(\lambda))] = \text{Var}[n^{1/2} a_n(S_1 + S_5)] \rightarrow \sigma_0^2(x_0, \lambda) \text{ as shown in}$$

lemma 5.2.4. So assuming $\sigma_0^2(x_0, \lambda) > 0$, it suffices to show that the array defining the T_n 's satisfies the Lindeberg condition, i.e. it suffices to show that for every $\epsilon > 0$

$$(5.3.4) \quad \frac{n}{\sigma_n^2} \iint_{A_n} (p_n(x, y) + q_n(x, y) - \mu(p_n) - \mu(q_n))^2 dH(x, y) \rightarrow 0, \text{ as } n \rightarrow \infty$$

where

$$(5.3.5) \quad \mu(p_n) = E(p_n(X, Y)), \quad \mu(q_n) = E(q_n(X, Y))$$

and

$$(5.3.6) \quad A_n = \{|p_n(x, y) + q_n(x, y) - \mu(p_n) - \mu(q_n)| > \epsilon \sigma_n\}.$$

Since σ_n^2 converges to a positive constant it suffices to prove that

$$(5.3.7) \quad g_n(\epsilon) = n \iint_{A_n} (p_n(x, y) + q_n(x, y) - \mu(p_n) - \mu(q_n))^2 dH(x, y)$$

goes to 0 as $n \rightarrow \infty$.

Now observe that

$$(5.3.8) \quad g_n(\epsilon) \leq 4n \iint_{A_n} p_n^2(x, y) dH(x, y) + 4n \iint_{A_n} q_n^2(x, y) dH(x, y) \\ + 4n\mu^2(p_n) + 4n\mu^2(q_n).$$

From (5.3.3) and (5.3.5) we have

$$\mu^2(p_n) = \frac{1}{na_n^2} \iiint K\left(\frac{G(x_1)-G(x_0)}{a_n}, \frac{F_{x_0}(y_1)-\lambda}{a_n}\right) K\left(\frac{G(x_2)-G(x_0)}{a_n}, \frac{F_{x_0}(y_2)-\lambda}{a_n}\right) \\ y_1 y_2 dH(x_1, y_1) dH(x_2, y_2).$$

But the integral above is of order $O(a_n^4)$, which may easily be seen by the usual transformation of variables (4.4.18) and consequently we have

$$(5.3.9) \quad n \cdot \mu^2(p_n) = O(a_n^2), \text{ which goes to } 0 \text{ as } n \rightarrow \infty.$$

Similarly

$$\mu^2(q_n) = \frac{(n-1)^2}{n^3 a_n^6} \iiint K^*\left(\frac{G(x_2)-G(x_0)}{a_n}\right) K^*\left(\frac{G(x_4)-G(x_0)}{a_n}\right) K_2\left(\frac{G(x_1)-G(x_0)}{a_n}, \frac{F_{x_0}(y_1)-\lambda}{a_n}\right) \\ K_2\left(\frac{G(x_3)-G(x_0)}{a_n}, \frac{F_{x_0}(y_3)-\lambda}{a_n}\right) y_1 y_3 I(y_2 \leq y_1) I(y_4 \leq y_3) dH(x_1, y_1) \\ dH(x_2, y_2) dH(x_3, y_3) dH(x_4, y_4),$$

and the last integral is exactly the same as that in (4.4.54) which was shown to be of order $O(a_n^7)$. Hence we have

$$(5.3.10) \quad n \mu^2(q_n) = O(a_n), \text{ which goes to } 0 \text{ as } n \rightarrow \infty.$$

Since

$$(5.3.11) \quad A_n \subset A_{1n} \cup A_{2n} \cup A_{3n}$$

where

$$A_{1n} = \{x, y: |p_n(x, y)| \geq \frac{\varepsilon \sigma_n}{3}\}$$

$$A_{2n} = \{x, y: |q_n(x, y)| \geq \frac{\varepsilon \sigma_n}{3}\}$$

$$A_{3n} = \{x, y: |\mu(p_n) + \mu(q_n)| \geq \frac{\varepsilon \sigma_n}{3}\},$$

we have

$$(5.3.12) \quad \int\limits_{A_n} p_n^2(x, y) dH(x, y) \leq \int\limits_{A_{1n}} np_n^2(x, y) dH(x, y) + \int\limits_{A_{2n}} np_n^2(x, y) dH(x, y) + \int\limits_{A_{3n}} np_n^2(x, y) dH(x, y).$$

Further

$$(5.3.13) \quad A_{1n} = \{x, y: \left| \frac{1}{n^{1/2} a_n} K\left(\frac{G(x) - G(x_0)}{a_n}, \frac{F_{x_0}(y) - \lambda}{a_n}\right) y \right| \geq \frac{\varepsilon \sigma_n}{3}\} \\ \subset \{x, y: |y| \geq \delta \sigma_n (n^{1/2} a_n)\} = B_{1n}, \text{ say,}$$

where $\delta = \frac{\varepsilon}{3M}$ and $M = \sup_{x, y} |K(x, y)|$.

Since $\sigma_n^2 \rightarrow$ positive constant, $n^{1/2} a_n \rightarrow \infty$, the sequence of sets

$$(5.3.14) \quad \{B_{1n}\} \rightarrow \emptyset, \text{ the empty set, as } n \rightarrow \infty.$$

Next, we have

$$A_{2n} = \{x, y: \left| \frac{n-1}{n^{3/2} a_n^3} K^*\left(\frac{G(x) - G(x_0)}{a_n}\right) \int\limits_{y_1}^y K_2\left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n}\right) y_1 I(y \leq y_1) dH(x_1, y_1) \right| \geq \frac{\varepsilon \sigma_n}{3}\}.$$

Since

$$\begin{aligned}
 & \left| \iint K_2' \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 I(y \leq y_1) dH(x_1, y_1) \right| \\
 & \leq \iint \left| K_2' \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) y_1 \right| dH(x_1, y_1) \\
 & = O(a_n^2),
 \end{aligned}$$

(which maybe easily seen by the routine transformation of variables explained by (4.4.13) and application of lemma 4.1.1) and K^* is bounded it follows that the sequence of sets

$$(5.3.15) \quad \{A_{2n}\} \rightarrow \phi \text{ as } n \rightarrow \infty.$$

Since both $\mu(p_n)$ and $\mu(q_n)$ go to 0 as $n \rightarrow \infty$, which may be seen from

(5.3.9) and (5.3.10), we also have that the sequence of sets

$$(5.3.16) \quad \{A_{3n}\} \rightarrow \phi \text{ as } n \rightarrow \infty.$$

Thus from (5.3.12) and (5.3.13) to (5.3.16) we obtain

$$\begin{aligned}
 (5.3.17) \quad \iint_{A_n} p_n^2(x, y) dH(x, y) & \leq \iint_{B_{1n}} p_n^2(x, y) + \iint_{A_{2n}} p_n^2(x, y) dH(x, y) \\
 & \quad + \iint_{A_{3n}} p_n^2(x, y) dH(x, y)
 \end{aligned}$$

and

$$(5.3.18) \quad n \iint_{A_n} q_n^2(x, y) dH(x, y) \leq \iint_{B_{1n}} n q_n^2(x, y) dH(x, y) + \iint_{A_{2n}} n q_n^2(x, y) dH(x, y) \\ + \iint_{A_{3n}} n q_n^2(x, y) dH(x, y).$$

where $\{B_{1n}\}$, $\{A_{2n}\}$ and $\{A_{3n}\}$ all $\rightarrow \phi$ as $n \rightarrow \infty$.

From (5.3.17), consider

$$\iint_{B_{1n}} n p_n^2(x, y) dH(x, y) = \frac{1}{a_n^2} \iint_{B_{1n}} K^2 \left(\frac{G(x) - G(x_0)}{a_n}, \frac{F_{x_0}^{-1}(y) - \lambda}{a_n} \right) y^2 dH(x, y).$$

By transformation of variables (4.4.13) and using Taylor expansions from lemma 4.1.1, we get

$$(5.3.19) \quad \iint_{B_{1n}} n p_n^2(x, y) dH(x, y) = \iint_{E_{1n}} K^2(u, v) (F_{x_0}^{-1}(\lambda))^2 du dv + O(a_n)$$

where $E_{1n} = \{u, v: |F_{x_0}^{-1}(\lambda + a_n v)| \geq \delta \sigma_n(n^{1/2} a_n)\}$.

Since K^2 is integrable and the set E_{1n} converges to ϕ , the integral on the R.H.S. of (5.3.19) goes to 0. The second term goes to zero since $a_n \rightarrow 0$ and consequently

$$\iint_{B_{1n}} p_n^2(x, y) dH(x, y) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The other two integrals on the R.H.S. of (5.3.17) can be handled exactly in the same manner and thus we have

$$(5.3.20) \quad n \iint_{A_n} p_n^2(x, y) dH(x, y) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

To show that $n \iint_{A_n} q_n^2(x, y) dH(x, y)$ goes to zero, we shall again deal with the

first integral on the R.H.S. of (5.3.18) only since the others can be handled on the very same lines.

Now

$$\begin{aligned}
 (5.3.21) \quad & n \iint_{B_{1n}} q_n^2(x, y) dH(x, y) \\
 &= \left(\frac{n-1}{n}\right)^2 \frac{1}{a_n} \iiint_{B_{1n}} K^{*2} \left(\frac{G(x) - G(x_0)}{a_n} \right) K_2' \left(\frac{G(x_1) - G(x_0)}{a_n}, \frac{F_{x_0}(y_1) - \lambda}{a_n} \right) \\
 &\quad K_2' \left(\frac{G(x_2) - G(x_0)}{a_n}, \frac{F_{x_0}(y_2) - \lambda}{a_n} \right) y_1 y_2 I(y \leq y_1) I(y \leq y_2) \\
 &\quad dH(x_1, y_1) dH(x_2, y_2) dH(x, y) \\
 &= \left(\frac{n-1}{n}\right)^2 \iiint_{E_{1n}} K^{*2}(u) K_2'(u_1, v_1) K_2'(u_2, v_2) I(v \leq v_1 \wedge v_2) (F_{x_0}^{-1}(\lambda))^2 \\
 &\quad du_1 dv_1 du_2 dv_2 dudv \\
 &\quad + o(a_n),
 \end{aligned}$$

where the last equality in (5.3.21) follows after the transformation of variables x, y, x_1, y_1, x_2, y_2 to u, v, u_1, v_1, u_2, v_2 as in (4.4.18), followed by the use of Taylor expansions from lemma 4.4.1. Since the set E_{1n} converges to ϕ as $n \rightarrow \infty$ and $a_n \rightarrow 0$ as $n \rightarrow \infty$, we have

$$n \iint_{B_{1n}} q_n^2(x, y) dH(x, y) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Hence we have

$$(5.3.22) \quad n \iint_{A_n} q_n^2(x, y) dH(x, y) \rightarrow 0, \text{ as } n \rightarrow \infty$$

and it now follows from (5.3.9), (5.3.10), (5.3.20), (5.3.22) and (5.3.8) that $g_n(\epsilon) \rightarrow 0$ as $n \rightarrow \infty$ for every $\epsilon > 0$. The proof of the Theorem is complete. $\square$

PROOF OF COROLLARY 5.1.1;

Observe that

$$E(I_1) = a_n^{-2} \iint K\left(\frac{G(x) - G(x_0)}{a_n}, \frac{F_{x_0}(y) - \lambda}{a_n}\right) dy dH(x, y) = \bar{Q}_{n, x_0}.$$

Hence from (4.5.20) we have

$$\bar{Q}_{n, x_0}(\lambda) - Q_{x_0}(\lambda) = O(a_n^2)$$

and consequently

$$(5.3.23) \quad n^{1/2} a_n [\bar{Q}_{n, x_0}(\lambda) - Q_{x_0}(\lambda)] = o(n^{1/2} a_n^3)$$

which goes to zero as $n \rightarrow \infty$ since $na_n^5 \rightarrow 0$. The Corollary is now a consequence of Theorem 5.1.1 and (5.3.23). $\square$

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